

TWISTORS, SPECIAL RELATIVITY, CONFORMAL SYMMETRY AND MINIMAL COUPLING - A REVIEW.

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Abstract

An approach to special relativistic dynamics using the language of spinors and twistors is presented. Exploiting the natural conformally invariant symplectic structure of the twistor space, a model is constructed which describes a relativistic massive, spinning and charged particle, minimally coupled to an external electro-magnetic field. On the two-twistor phase space the relativistic Hamiltonian dynamics is generated by a Poincaré scalar function obtained from the classical limit (appropriately defined by us) of the second order, to an external electro-magnetic field minimally coupled, Dirac operator. In the so defined relativistic classical limit there are no Grassman variables. Besides, the arising equation that describes dynamics of the relativistic spin differs significantly from the so called Thomas Bergman Michel Telegdi equation.

1 INTRODUCTION.

In relativistic physics one talks often about Lorentz four-vectors and Lorentz four-tensors. In this introduction we wish to analyze the mathematical meaning of these notions in a narrative and formula-free way. Still we would like to keep a certain degree of strictness. Formulas making our attempts a little more formal will appear in the next sections.

Lorentz four-vectors and Lorentz four-tensors are vectors and tensors in the usual *vector space* \mathbb{R}^4 equipped with the pseudo-Euclidean (Minkowski) metric η of signature $(+---)$. Besides of being vectors and tensors in \mathbb{R}^4 , they are also invariant geometrical objects with respect to the action of the homogeneous Lorentz group preserving η . It is very well-known that, because of the pseudo-Euclidity of η , the vectors in the vector space \mathbb{R}^4 are divided into three types: time-like (their Minkowski norms are positive), space-like (their Minkowski

norms are negative) and null-like (their Minkowski norms are zero). The null-like vectors are very special and, as mentioned above, are invariant geometrical objects with respect to the homogeneous Lorentz group. However they are also invariant with respect to a change of scale, i.e. prolongations or shortenings of a null-like vector does not change its vanishing Minkowski norm.

The connected component of the identity of the homogeneous Lorentz group, regarded as a six dimensional smooth manifold, is not simply connected but covered by a group (simply connected six dimensional smooth manifold) that can be represented by the matrix group¹ $SL(2, \mathbb{C})$. The connected component of the identity of the homogeneous Lorentz group becomes just a two to one homomorphic image of $SL(2, \mathbb{C})$. Two matrices in $SL(2, \mathbb{C})$, one with plus and one with minus sign, represent the same element of the identity connected component of the homogeneous Lorentz group.

$SL(2, \mathbb{C})$ matrices act naturally on complex vectors in \mathbb{C}^2 . The group $SL(2, \mathbb{C})$ preserves an antisymmetric form (a “metric”) ϵ in \mathbb{C}^2 . The complex vectors in \mathbb{C}^2 regarded as geometrical objects with respect to ϵ are called (Weyl) spinors and \mathbb{C}^2 equipped with ϵ is called the (Weyl) spinor space S . An entire $SL(2, \mathbb{C})$ invariant tensor algebra over S (and its complex conjugate counterpart) arises in this way.

It turns out then that null-vectors in the Minkowski vector space may be regarded as simple hermitian spinor tensors of second rank in this tensor algebra over S and its complex conjugate, i.e. each spinor together with its complex conjugate counterpart defines a null-vector. More exactly each null-vector in Minkowski vector space is defined by a unique set of spinors in S given by a spinor modulo its multiplication by a complex number whose absolute value is equal to one i.e. modulo multiplication by a phase factor. Although spinors are simplest geometrical objects (complex vectors) in S their interpretation in terms of geometrical objects in the Minkowski vector space is not so simple. For example, the phase factor (angle) of a spinor has an exact but quite complicated geometrical meaning² in the Minkowski vector space. Spinors modulo multiplication by a non-zero complex number³ represent null-direction in the Minkowski vector space, i.e. represent null-vectors modulo their multiplication by a non-zero real number. This and much much more is carefully described in, for example, Penrose’s and Rindler’s book [20].

In this review, Lorentz four-vectors and four-tensors (and thereby Minkowski vector space itself) will be treated as a subset of the ($SL(2, \mathbb{C})$ invariant) spinor tensor algebra over S and its complex conjugated counterpart. Minkowski vector space is thus regarded as less elementary than \mathbb{C}^2 , the complex vector space equipped with the “metric” ϵ , see [26] chapter 2 and Appendix A.

One of the remarkable insights that follows from such point of view and nowadays used extensively in calculations within general relativity (see for example, [17, 20, 26]), is that any orthonormal basis (three space-like directions and one time-like direction) with respect to the pseudo-Euclidean metric $(+ - - -)$ in the Minkowski vector space may be regarded as constructed out of just any pair of non-parallel (non-proportional) spinors (i.e. a pair

¹ 2×2 matrices with complex entries and determinants equal to one.

²it describes an angle variable associated with the so called flag of a spinor, see developments in the next section and e.g.[20].

³in other words complex lines through the origin in \mathbb{C}^2 defining all points on $\mathbb{CP}(1)$ (complex projective space of dimension one), i.e. on the Riemann sphere.

of spinors with non-vanishing “scalar product” with respect to the ϵ “metric”) that are normalised to one (two such spinors are said to form a spin frame. Note that ϵ -norm of any spinor is always equal to zero).

More concrete developments will follow in the sequel. But first let us make some remarks on the *difference* between the *Minkowski vector space* as referred to above and the *affine Minkowski space* representing the space of physical events. Let us then discuss briefly how this might lead us, in a natural way, to the introduction of the twistor space.

The Lorentz tensor algebra over the *Minkowski vector space* is not sufficient for the description of (special) relativistic physical phenomena. An upgrading of the Minkowski vector space to the *affine Minkowski space* is needed. The points in the affine Minkowski space represent a postulated geometrical continuum (?) of physical events. Events are not Lorentz four-vectors although they may be identified and represented by *position* four-vectors. It is essential to be aware of the difference between these two geometrical objects. In practical applications an origin is always chosen to start with, so this distinction is not always so transparent. On the other hand, the four-intervals between two arbitrary events in space-time are, of course, Lorentz four-vectors.

The invariance group of the affine Minkowski space (compared with the Lorentz group invariance of just the Minkowski four-vector space) is now extended by the arbitrariness of the choice of the space-time origin. The composition of the homogeneous Lorentz group with the group of translations of the origin in space-time⁴ is called the Poincaré group or sometimes the inhomogeneous Lorentz group⁵. At each point (event) in the affine Minkowski space, the space of the Lorentz invariant four-intervals pointing towards all other points of the affine Minkowski space form a Minkowski vector space, i.e. they form a Minkowski tangent space at that point. As mentioned above, to label an event in space-time a new notion is needed, namely, the notion of a *position four-vector*. Such position four-vectors behave like usual four-vectors under the action of the Lorentz group but in contrast to genuine four-vectors, they are also affected by changes of the space-time origin, i.e. by space-time translations.

By analogy with the special role played by null-vectors in the Minkowski vector space (effectively only one spinor modulo its phase defines such a null-vector) straight null-lines in the Minkowski space play also an exceptional role. These null-lines may be identified with possible trajectories of free massless and spinless particles carrying linear and angular (with respect to some arbitrarily chosen space-time origin) four-momenta. The pseudo-Euclidean norms of four-intervals between any two arbitrary space-time points along such a null-line equal zero. This implies that multiplying genuine null-vectors at each space-time point by a non-zero real number (different at each space-time point) does not affect the null-lines (passing through this point and) having these null-vectors as directions at this point. In order to keep the straight null-lines in Minkowski space unaffected it is therefore sufficient to require that the Minkowski metric is preserved only modulo its multiplication by an arbitrary non zero positive real valued function on the Minkowski space⁶. Thus the

⁴note that these two subgroups, i.e. the translations and the Lorentz transformations do not commute in general.

⁵while the Lorentz group manifold is six dimensional (six parameters), the Poincaré group manifold is ten dimensional.

⁶it may be shown that in the Minkowski space this is equivalent to the requirement that the Lorentz

set of all straight null-lines in the Minkowski space is invariant with respect to an even larger symmetry group than the symmetry (Poincaré) group of the Minkowski space itself. A (non-linear and local) representation in the Minkowski space itself of this enlarged fifteen dimensional conformal symmetry group has been known for quite a long time and is denoted by $C(1,3)$. The first ten dimensions of its Poincaré subgroup (manifold) have a clear physical meaning. They represent Lorentz transformations (rotations and the so called boosts) and the four-translations. The fundamental laws of nature, stating that the angular momentum and the velocity of the centre of the energy of an isolated physical system are always conserved, may be derived from the Lorentz symmetry while the system's conservation of its energy and its linear momentum, may be derived from the translational symmetry. The remaining symmetry concealed in the five additional parameters of the conformal group has (in terms of conservation laws) no clear physical meaning at the moment. There is a lot of subtle points concerning this enlarged (conformal) invariance group. Many of them will not be discussed here, for example, we will not discuss the fact that the action of this enlarged invariance conformal group is non-linear and only local while represented in the Minkowski space and what this possibly implies, etc. The reader may consult original papers on this topic [18, 19, 20] (and references therein) for a detailed discussion of the issue. In this review we will focus only on certain aspects that will lead us into models describing dynamics of relativistic spinning massive particles in terms of the so called twistors.

What are twistors? They are related to the conformal group $C(1,3)$ mentioned above in the following fashion. A linear and also four to one covering group (manifold) of the group $C(1,3)$ may be represented by the matrix group⁷ $SU(2,2)$. The set of complex valued vectors in \mathbb{C}^4 equipped with a pseudo-hermitian form g preserved by the action of the $SU(2,2)$ group⁸ is called the twistor space T of non-projective twistors $\{Z, W, \dots\}$. From this it should follow that (the covering (manifold) of the connected component of the identity of) the Poincaré group is a subgroup of the $SU(2,2)$ group.

It is then possible to identify certain $SU(2,2)$ and/or Poincaré invariant/covariant functions on T with geometrical and dynamical/kinematical variables of a (classical limit of a) massless spinning object in the Minkowski space. Using at least two copies of T , certain $SU(2,2)$ and/or Poincaré invariant/covariant geometrical and dynamical/kinematical variables (including the position four-vectors) of a (classical limit of a) charged massive spinning object in the Minkowski space may be also identified.

The imaginary part of the pseudo-hermitian form g in \mathbb{C}^4 constitutes an $SU(2,2)$ invariant symplectic structure on T which allows the non-projective twistor space to be treated as the simplest possible (extended) phase space of a (classical limit of a) massless spinning object equipped with globally defined and canonically conjugated conformally and thereby also Poincaré invariant/covariant variables.

The subset of twistors (modulo multiplication by non-zero complex numbers with their scalar product of any two non-null Minkowski intervals divided by the product of their norms should be unaffected hence the name conformal invariance; “the Minkowski angles” are to be preserved.

⁷special (determinants equal to one) unitary matrices with complex valued entries preserving the pseudo-hermitian form g of signature $++--$.

⁸in order to see explicitly and keep track of how the (spinorial versions of) the Poincaré and Lorentz groups are “inbedded” inside the $SU(2,2)$ as subgroups it is necessary to use special spinor representations of g and $SU(2,2)$. See developments in the sequel.

absolute value equal to one, i.e. modulo multiplication by phase factors), in the twistor space $T \simeq \mathbb{C}^4$, with their pseudo hermitian norms equal to zero (null-twistors) may, when interpreted in the Minkowski space, be identified with the set of all possible straight trajectories of massless spinless particles with given linear null- and (null-) orbital angular four-momenta that with respect to the above mentioned symplectic structure fulfill the Poisson bracket commutation relations of the Poincaré algebra. If the pseudo-hermitian norm is not equal to zero, then the corresponding twistors may still be identified with massless spinning objects in the Minkowski space with given linear null- and (null-) angular four-momenta that again fulfill the commutation relations of the Poisson bracket Poincaré algebra with respect to the symplectic structure defined by the imaginary part of the pseudo-hermitian form. Remarkably, such (non-quantum, i.e. classical limits of) massless spinning objects do not have well-defined trajectories when one tries to interpret them in the (real) Minkowski space⁹. However, taking into account the phase space structure of T , pairs of non-coinciding twistors (or quite generally any number (greater than two) of non-coinciding twistors) may also be used to define dynamical variables representing four-positions [4, 5, 9, 19] of charged massive and, in general, spinning object in the Minkowski space. This remarkable fact¹⁰ will be utilised extensively by us in this review.

Summarizing, the description of (massless non-local) geometrical objects, “living” in the Poincaré invariant Minkowski space, in terms of abstract conformally invariant (local) geometrical objects “living” in the twistor space, uses the fact that the conformal group representation $C(1, 3)$ of the Minkowski space is a 4-1 homomorphic image of the complex matrix group $SU(2, 2)$ acting on the twistor space $T \simeq \mathbb{C}^4$. Any element in $C(1, 3)$ may be represented by a matrix A in $SU(2, 2)$ or by $-A$ or by iA or finally by $-iA$. $C(1, 3)$ is thus a homomorphic image of $SU(2, 2)$. Twistors with norms equal to zero are identified with null-lines in the Minkowski space. Non-null twistors are geometrically identified with Robinson congruences. Any geometrical object “living” in the Minkowski space may, in a relatively easy way, be described in terms of geometrical objects (twistor-tensors) in the twistor space but not necessarily, and not so easily, in the opposite direction.

To be somewhat more exact in the definition of twistors let us make certain additional remarks, in risk of being repetitive of what already has been said above. The complex vectors in \mathbb{C}^4 equipped with a “metric” (pseudo-hermitian form g) preserving $SU(2, 2)$ are, in fact, called *non-projective* twistors. The pseudo-hermiticity implies that the norm of a non-projective twistor in T may assume positive, null or negative real values. Non-projective twistors having their norms equal to zero are called non-projective, null-twistors as already mentioned. The non-projective twistors with positive norms are called non-projective, positive helicity twistors. The non-projective twistors in T with negative norms are called non-projective, negative helicity twistors. Non-projective twistors modulo multiplication by non-zero complex numbers (changing the value of its norm but not the sign) are called *projective* twistors and form a space of complex lines through the origin in the complex vector space \mathbb{C}^4 . The set of complex lines in \mathbb{C}^4 forms a six dimensional manifold (with

⁹instead of trajectories they may be interpreted as twisting Robinson congruences of null-lines filling up the entire Minkowski space. When the twist (i.e. the norm of the corresponding twistor) vanishes such a congruence will meet, and thereby, in fact, define a unique null-line previously already identified as the null-line represented by this null-twistor [16, 18, 19, 20].

¹⁰remarkable because non-local massless objects in Minkowski space define explicitly an event (extremely local object) in space-time. Events (local objects) become secondary, while non-local massless objects (in general represented by Robinson congruences) are primary in such an approach.

topology S^7/S^1) and is denoted by $\mathbb{CP}(3)$. Note that the pseudo-Euclidean norm in T splits $\mathbb{CP}(3)$ into three parts depending on the sign $(+, 0, -)$ of the corresponding non-projective twistor representatives. Thus, T is the space of homogeneous coordinates of $\mathbb{CP}(3)$ or, if one so wishes to identify it, the complex line bundle over $\mathbb{CP}(3)$.

It has been shown by us [4, 5, 28, 29, 30] that twistors and the symplectic structure defined by the imaginary part of the pseudo-hermitian form preserved by $SU(2, 2)$ define an extended phase space of massive, electrically charged and in general spinning objects. Among the relativistic dynamical/kinematical variables describing these objects, the Minkowskian space-time positions are singled out by the formalism. In the simplest case, the four-positions of the objects are *identified* as certain Poincaré covariant functions of two non-coinciding twistors. Physical events become in this way (at least mathematically) secondary objects, *defined* by twistor variables. This way of looking at space-time events is analogous (but somewhat more subtle) to the way one regards any time-like or space-like Lorentz four-vector as constructed from a pair of non-proportional spinors in the spinor space (see e.g. [26] chapter 2 and Appendix A).

The relativistic dynamics of a (classical limit of a) massive and spinning particle-object as mentioned above, may be viewed as a canonical flow generated by an appropriately chosen real valued function (e.g. by identifying the classical limit of the second order minimally coupled Dirac operator as such a function) on a direct product of two copies¹¹ of T where the relativistically invariant canonical symplectic structure consists of a direct sum of two copies of the imaginary parts of the pseudo-hermitian form preserved by the action of $SU(2, 2)$ [1, 3]. We now proceed to make all the statements above more concrete, mathematically. We will be very sloppy with proofs because we wish only to present the known results and use them promptly for our purposes, namely, in order to derive from the formalism the relativistic dynamical equations describing a massive, charged and spinning “particle” moving in an external electro-magnetic field.

2 SPINORS, LORENTZ FOUR-VECTORS AND LORENTZ FOUR-TENSORS.

In this section we start with the (abstract) vector space \mathbb{C}^2 equipped with a $SL(2, \mathbb{C})$ invariant antisymmetric “metric” ϵ . Such a two-dimensional complex vector space is called the Weyl spinor¹² space S and will be used to construct the physical (less abstract) notion of the Minkowski vector space M_v , i.e., a real vector space \mathbb{R}^4 equipped with the Minkowski metric η which is invariant with respect to the $SO(1, 3)$ group representing the (identity connected) homogeneous Lorentz group. As a consequence, the Lorentz tensor algebra will appear as a subset of the complex valued spinor-tensor algebra (see e.g. [17, 20]).

We use greek lower case letters to denote spinors and spinor-tensors in S . The greek lower case letter ϵ will be reserved to denote the $SL(2, \mathbb{C})$ invariant antisymmetric “metric” tensor in \mathbb{C}^2 . Note that for each spinor in S there exists its complex conjugate counterpart. Besides, for each spinor there exists also its covariant counterpart (with respect to the $SL(2, \mathbb{C})$ ($\bar{SL}(2, \mathbb{C})$) invariant antisymmetric “metric” ϵ ($\bar{\epsilon}$)). Thus, each contravariant spinor has three “brothers”, its covariant version, its complex conjugate, and the covari-

¹¹less the diagonal points in that product.

¹²we will in most cases just write “spinor” omitting the name of Weyl.

ant version of its complex conjugate. We will distinguish among them according to the following well-known convention: the contravariant spinors will be distinguished by latin upper case superscript letters taking values 0 and 1 with respect to some spin frame (i.e. a normalised spinor basis). If no frame is chosen, the index just tells us what kind of entity we are dealing with (Penrose's abstract index notation [17]). The covariant spinors will be distinguished by latin upper case subscript letters taking values 0 and 1 with respect to such a spin frame. Complex conjugation will be denoted by a bar over the symbol with simultaneous priming of the subscript and superscript letters. According to this convention we will denote the spinor space and the corresponding complex conjugate spinor space by

$$S = (\mathbb{C}^2, \epsilon_{AB}), \quad \bar{S} = (\bar{\mathbb{C}}^2, \bar{\epsilon}_{A'B'}). \quad (2.1)$$

In addition, we will have the covariant versions of these spaces that will be denoted by

$$S^* = (\mathbb{C}^2, \epsilon^{AB}), \quad \bar{S}^* = (\bar{\mathbb{C}}^2, \bar{\epsilon}^{A'B'}), \quad (2.2)$$

where ϵ^{AB} is the inverse of ϵ_{BA} while $\bar{\epsilon}^{A'B'}$ is the inverse of $\bar{\epsilon}_{B'A'}$ ¹³. The invariance of the ϵ -“metric” may now be expressed as follows

$$\epsilon_{AB} = L_A^C L_B^D \epsilon_{CD}, \quad \bar{\epsilon}_{A'B'} = \bar{L}_{A'}^{C'} \bar{L}_{B'}^{D'} \bar{\epsilon}_{C'D'}, \quad (2.3)$$

where

$$L_A^C \in \text{SL}(2, \mathbb{C}) \quad \bar{L}_{A'}^{C'} \in \overline{\text{SL}(2, \mathbb{C})}. \quad (2.4)$$

With respect to a spin frame in \mathbb{C}^2 we may use the following numerical representations of the antisymmetric “metric” preserved by the action of $\text{SL}(2, \mathbb{C})$ and of $\overline{\text{SL}(2, \mathbb{C})}$ on \mathbb{C}^2 and on its complex conjugate $\bar{\mathbb{C}}^2$:

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{\epsilon}_{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.5)$$

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{\epsilon}^{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.6)$$

Thus, the antisymmetric “metrics” ϵ and $\bar{\epsilon}$ preserved by $\text{SL}(2, \mathbb{C})$ and $\overline{\text{SL}(2, \mathbb{C})}$ are also symplectic structures on \mathbb{C}^2 and on $\bar{\mathbb{C}}^2$ respectively. Consequently the “norm” of any spinor is, by definition, always equal to zero ($\epsilon_{AB}\pi^A\pi^B = 0$ for any π in S).

Any element in $\text{SL}(2, \mathbb{C})$ and in its complex conjugate $\overline{\text{SL}(2, \mathbb{C})}$ is, with respect to a spin frame, represented by the following set of two by two matrices with complex entries, matrices having their determinants equal to one¹⁴:

¹³so that we always have $\epsilon^{CB}\epsilon_{BD} = \delta_D^C$, i.e. mappings from S to S^* (or from \bar{S} to \bar{S}^*) correspond to a lowering of the “contravariant” spinor index towards the “covariant” index nearest to the kernel letter ϵ (or $\bar{\epsilon}$) while the inverse mapping is a raising of a “covariant” spinor index toward the second contravariant index in the kernel letter ϵ (or $\bar{\epsilon}$), a good mnemonic rule.

¹⁴the four complex valued entries in the matrix will then represent just six real variables namely the six continuous parameters of the Lorentz group which will be seen more clearly later on in this review.

$$L_N^M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}), \quad (2.7)$$

$$\bar{L}_{N'}^{M'} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in \overline{SL(2, \mathbb{C})}, \quad (2.8)$$

$$l_M^N := (L^{-1})_M^N = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL(2, \mathbb{C}), \quad (2.9)$$

$$\bar{l}_{M'}^{N'} := (\bar{L}^{-1})_{M'}^{N'} = \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix} \in \overline{SL(2, \mathbb{C})}. \quad (2.10)$$

The invariance of the ϵ -“metric” in (2.3) may be now, with respect to a spin frame, expressed in matrix form as:

$$L^T \epsilon L = \epsilon, \quad \bar{L}^T \bar{\epsilon} \bar{L} = \bar{\epsilon}, \quad (2.11)$$

where the superscript T denotes transposition of a matrix.

A simple exercise is to verify identities in (2.11) by using the matrix representations in (2.5)–(2.8).

According to the above described convention, whether the spin frame is chosen or not, any contravariant spinor, its covariant version and their complex conjugate counterparts may be explicitly written as

$$\pi^A \in S, \quad \pi_B := \pi^A \epsilon_{AB} \in S^*, \quad \bar{\pi}^{A'} \in \bar{S}, \quad \bar{\pi}_{B'} := \bar{\pi}^{A'} \bar{\epsilon}_{A'B'} \in \bar{S}^*, \quad (2.12)$$

and conversely:

$$\pi^A = \pi_B \epsilon^{AB} \in S, \quad \bar{\pi}^{A'} := \bar{\pi}_{B'} \bar{\epsilon}^{A'B'} \in \bar{S}. \quad (2.13)$$

Note the order of contractions, i.e. to obtain a covariant spinor contraction with the first ϵ subscript letter is performed, while a contravariant spinor is created by contraction with the second superscript letter of the ϵ -“metric”.

Obviously, action of an $SL(2, \mathbb{C})$ transformation on a spinor reads

$$\tilde{\pi}^M = L_N^M \pi^N, \quad \tilde{\pi}^{M'} = \bar{L}_{N'}^{M'} \bar{\pi}^{N'}, \quad \tilde{\pi}_M = l_M^N \pi_N, \quad \tilde{\pi}_{M'} = \bar{l}_{M'}^{N'} \bar{\pi}_{N'}. \quad (2.14)$$

Consider now all hermitian ($v^{AA'} = \bar{v}^{A'A}$) mixed (i.e., once unprimed and once primed) spinor-tensors of second rank given by

$$v^{AA'} := a \pi^A \bar{\pi}^{A'} + u \pi^A \bar{\eta}^{A'} + \bar{u} \eta^A \bar{\pi}^{A'} + b \eta^A \bar{\eta}^{A'}, \quad (2.15)$$

where π and η are two non-proportional spinors (i.e. $\epsilon_{AB}\pi^A\eta^B \neq 0$), a, b arbitrary real numbers and u an arbitrary complex number. The set of such hermitian mixed spinor-tensors will be denoted by $(S \otimes \bar{S})_h$. The set of all mixed spinor-tensors of rank two¹⁵ will be denoted by $S \otimes \bar{S}$.

From (2.14) it now follows that any mixed spinor-tensor of second rank and therefore, any hermitian one, under the action of an $SL(2, \mathbb{C})$ transformation, is transformed as follows

$$\tilde{v}^{AA'} = L_B^A \bar{L}_{B'}^{A'} v^{BB'}. \quad (2.16)$$

The subgroup $((SL(2, \mathbb{C}) \otimes \overline{SL(2, \mathbb{C})})_h$ of hermitian transformations in $SL(2, \mathbb{C}) \otimes \overline{SL(2, \mathbb{C})}$ of the type $L_B^A \bar{L}_{A'}^{B'}$ as in (2.16) acting in the space of mixed spinor-tensors of second rank such as in (2.15) is, as well-known, *isomorphic* to the group¹⁶ $SO(1, 3)$. By definition, the group of hermitian transformations in $SL(2, \mathbb{C}) \otimes \overline{SL(2, \mathbb{C})}$ preserves the hermitian tensor product of the ϵ -metrics,

$$\eta_{AA'BB'} = L_A^M \bar{L}_{A'}^{M'} L_B^K \bar{L}_{B'}^{K'} \eta_{MM'KK'}, \quad (2.17)$$

where

$$\eta_{AA'BB'} := \epsilon_{AB} \bar{\epsilon}_{A'B'}. \quad (2.18)$$

Because of the above mentioned isomorphism we may identify the group of hermitian transformations $(SL(2, \mathbb{C}) \otimes \overline{SL(2, \mathbb{C})})_h$ of the type displayed in (2.16) with the group $SO(1, 3)$ representing the identity component of the homogenous Lorentz group. This, in turn, implies that M_v may be identified with $(S \otimes \bar{S})_h$ and that $\epsilon \bar{\epsilon}$ may be identified with the Minkowski metric η as displayed in (2.18). For each (abstract) Lorentz four-vector v^a we thus have the identification

$$v^a = v^{AA'} = \bar{v}^{A'A}. \quad (2.19)$$

For the Lorentz transformations in M_v , the above mentioned isomorphism implies the following identifications

$$L_n^m = L_{NN'}^{MM'} = L_N^M \bar{L}_{N'}^{M'}, \quad (L^{-1})_m^n := l_M^N \bar{l}_{M'}^{N'} \quad \text{where} \quad L_n^m \in SO(1, 3). \quad (2.20)$$

The Minkowski metric η now appears as a fourth order hermitian tensor product of the ϵ “metric” and $\bar{\epsilon}$ “metric” (see (2.18)),

$$\eta_{ab} = \eta_{AA'BB'}. \quad (2.21)$$

Referring to the above identification, one could say that the symplectic structures ϵ and $\bar{\epsilon}$ on the two spinor spaces S and \bar{S} represent “square roots” of the Minkowski metric η

¹⁵with a, b in (2.15) being now arbitrary complex number and the complex numbers u, \bar{u} not being necessarily complex conjugate to each other.

¹⁶preserving the metric η of signature $(+ - - -)$ in \mathbb{R}^4 .

while the two spinor spaces themselves represent “square roots” of the M_v (Minkowski *vector* space) itself.

It should be understood that exact relations rendering coordinates of Minkowski four-tensors in terms of coordinates of spinor tensors¹⁷ depend on the choice of the explicit isomorphism $\sigma_{AA'}^a, \sigma_{BB'}^b$ between $(SL(2, \mathbb{C}) \otimes \overline{SL(2, \mathbb{C})})_h$ and $SO(1, 3)$, where $\sigma_{AA'}^a$ defines an explicit bijective mapping between mixed contravariant spinors of second rank and contravariant Lorentz vectors¹⁸. The isomorphism represented by $\sigma_{AA'}^a$ is sometimes called the Infeld-Van der Waerden connecting quantity. With respect to a spin frame in the spin vector space and with respect to an ortho-normal frame in the Minkowski vector space, the four two-dimensional matrices $\sigma_{AA'}^a$ are simply given by the two-dimensional (chirally reduced) Dirac matrices.

A convenient and frequently used explicit choice of the $\sigma_{AA'}^a$ matrices is provided by three, two by two, dimensional Pauli matrices and one two by two dimensional identity matrix, all four matrices multiplied by $\frac{1}{\sqrt{2}}$. This extra numerical factor is inserted in order to harmonize normalisation of any pair of non-parallel spinors with respect to the “metric” ϵ (and/or $\bar{\epsilon}$) with ortho-normalisation of the corresponding Lorentz four-vectors (see (2.15)) with respect to the arising pseudo-Euclidean metric η in (2.21) in M_v . However, in this review we shall not need explicit coordinate expressions very often. All necessary details we are omitting here may be found in Penrose’s and Rindler’s book [20] (see also [26] chapter 2).

While defining Lorentz four-tensors of various ranks in terms of hermitian spinor-tensors, it is extremely useful to note that only the symmetric part of non-mixed (i.e. either primed or non-primed) spinor-tensors is important, the antisymmetric part reduces itself to an $SL(2, \mathbb{C})$ (or $\overline{SL(2, \mathbb{C})}$) spinor-tensor contraction times the ϵ (or $\bar{\epsilon}$) “metric”. This is a consequence of the almost obvious (Fierz) identity:

$$\pi_A \eta_B - \pi_B \eta_A = \epsilon_{CD} \pi^C \eta^D = \pi_D \eta^D \epsilon_{AB}. \quad (2.22)$$

Another important property of the spinor-tensor algebra is that any symmetric spinor tensor is always simple, i.e., of the form

$$\nu^{(ABCD \dots F)} = \alpha^{(A} \beta^B \gamma^C \zeta^D \dots t^F), \quad (2.23)$$

where round brackets, as usual, denote symmetrisation and the spinors defining ν in (2.23) are only unique up to multiplication by a constant nonzero complex number. Therefore, each such spinor defines a null-direction (not an entire null four-vector) in the Minkowski vector-space. Some of these directions may coincide. In this case, one says that ν is algebraically special.

For example, the identity in (2.22) and the property described in (2.23) imply that the antisymmetric Lorentz four-tensor of second rank, expressed as an hermitian spinor-tensor of fourth rank, may be written as follows

$$M^{ab} = -M^{ba} = M^{AA'BB'} = -M^{BB'AA'} = \mu^{(AB)} \epsilon^{A'B'} + \bar{\mu}^{(A'B')} \epsilon^{AB}, \quad (2.24)$$

¹⁷coordinates require, of course, a choice of a spin frame (normalised spinor basis) in the spinor space S .

¹⁸possibly complexified if the mixed spinor tensor of second rank is not hermitian.

where circle brackets denote symmetrization,

$$\mu^{(AB)} := \frac{1}{2} \epsilon_{A'B'} (M^{AA'BB'} + M^{BA'AB'}), \quad (2.25)$$

and where

$$\mu^{(AB)} = \zeta^{(A} \iota^{B)}, \quad (2.26)$$

for some spinors ζ^A and ι^B .

From (2.16) one notes also that a simple spinor π , modulo multiplication by a phase factor, defines a Lorentz four-vector P with a Lorentz norm equal to zero, i.e. a Lorentz null four-vector (not just a null-direction)

$$P^n := \pi^N \bar{\pi}^{N'}. \quad (2.27)$$

To obtain Lorentzian interpretation of the spinor phase one observes then that a simple spinor also defines a unique (but algebraically special) antisymmetric Lorentz four-tensor of second rank

$$P^{ab} = -P^{ba} = P^{AA'BB'} = -P^{BB'AA'} = \pi^A \pi^B \epsilon^{A'B'} + \bar{\pi}^{A'} \bar{\pi}^{B'} \epsilon^{AB}. \quad (2.28)$$

P^a in (2.27) is called the flagpole of a (simple) spinor while P^{ab} in (2.28) is known as its flagplane. While a spinor defines its flagpole and its flagplane uniquely (in a “quadratic fashion”), from a given flagpole and its (algebraically special) flagplane the corresponding spinor can be recovered only up to a sign. This indicates again that the two dimensional spinors are more elementary than Lorentz four-vectors.

More details about how to manufacture Lorentz four-tensors out of spinor-tensors using rules of the spinor algebra can be found, for example, in Penrose’s and Rindler’s book [20] (see also [26] chapter 2).

Spinors may also be introduced using the so called Clifford algebra approach but the fundamental role of spinors in such an approach is no longer as transparent as was outlined above. On the other hand, the Clifford algebra approach may be easily generalised. It becomes possible to construct spinors for real metric vector spaces of any dimension and any non-degenerate signature¹⁹. For more details about Clifford algebra constructions see, for example, the appendix in the second volume of Penrose’s and Rindler’s book [20].

3 TWISTORS, EVENTS AND PARTICLES.

In this section the notion of the spinor space will be extended and generalised in such a way so that it will become a four dimensional complex vector space \mathbb{C}^4 equipped with a “metric” (a pseudo-hermitian form), invariant with respect to the action of the $SU(2, 2)$ matrix group. This group contains a subgroup composed of elements of the form $L \oplus \bar{L}$,

¹⁹according to Professor Rafał Abłamowicz, the Clifford algebra and its spinors may be introduced even when the metric is degenerate.

$\bar{l} \equiv \bar{L}^{-1}$ that belong to the matrix group in the direct sum of $\text{SL}(2, \mathbb{C})$ and $\overline{\text{SL}(2, \mathbb{C})}$, i.e. in $\text{SL}(2, \mathbb{C}) \oplus \overline{\text{SL}(2, \mathbb{C})}$ (see (3.5)). We will start from this subgroup, which represents the covering of the identity component of the Lorentz group, and successively fills up to the $SU(2, 2)$ matrix group. By using this fact we wish to show how the remaining parts of $SU(2, 2)$ correspond to four-translations, dilations and special conformal transformation in the *affine* Minkowski space. The complex vectors $\{Z^\alpha, W^\alpha, \dots\}$ of the complex vector space \mathbb{C}^4 equipped with $SU(2, 2)$ invariant pseudo-hermitian form are called (non-projective) twistors. Twistors are then geometrical vectors with respect to the group $SU(2, 2)$. To keep track of the physical interpretation it is essential to represent a twistor (although not $SU(2, 2)$ invariantly but only “Poincaré” covariantly and “Lorentz” ($\text{SL}(2, \mathbb{C})$) invariantly) in terms of two spinors. One consequence of this is that the pseudo-hermitian form (of signature $++--$) preserved by $SU(2, 2)$ will not be diagonal in such a representation.

First consider a direct sum of two spinor spaces $T := S \oplus \bar{S}^* \simeq \mathbb{C}^4$. From the analysis in the previous section it follows directly that, geometrically, T is invariant with respect to the above mentioned subset of $\text{SL}(2, \mathbb{C}) \oplus \overline{\text{SL}(2, \mathbb{C})}$ matrix transformations. More explicitly, the action of the “Lorentz” group on an element in $S \oplus \bar{S}^*$ reads

$$\begin{pmatrix} \tilde{\omega}^B \\ \tilde{\pi}_{B'} \end{pmatrix} = \begin{pmatrix} L_A^B & 0 \\ 0 & \bar{l}_{B'}^{A'} \end{pmatrix} \begin{pmatrix} \omega^A \\ \pi_{A'} \end{pmatrix}, \quad (3.1)$$

where $Z^\alpha = \begin{pmatrix} \omega^A \\ \pi_{A'} \end{pmatrix}$ represents a vector in T .

The block diagonal transformations²⁰ in (3.1) preserve, of course, the blockdiagonal “metric”

$$\begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & \bar{\epsilon}^{B'A'} \end{pmatrix} \quad (3.2)$$

or, more explicitly,

$$\begin{pmatrix} L_A^B & 0 \\ 0 & \bar{l}_{B'}^{A'} \end{pmatrix} \begin{pmatrix} \epsilon_{BC} & 0 \\ 0 & \bar{\epsilon}^{C'B'} \end{pmatrix} \begin{pmatrix} L_D^C & 0 \\ 0 & \bar{l}_{C'}^{D'} \end{pmatrix} = \begin{pmatrix} \epsilon_{AD} & 0 \\ 0 & \bar{\epsilon}^{D'A'} \end{pmatrix}. \quad (3.3)$$

The elements in $T := S \oplus \bar{S}^* \simeq \mathbb{C}^4$ acted upon by transformations of the form (3.1), are geometrical objects with respect to the “metric” in (3.2), called **Dirac bispinors**. Resuming, we note that the splitting of T into a direct sum of two spinor spaces $S \oplus \bar{S}^*$ is invariant with respect to the action of the (universal covering of the) identity component of the homogeneous Lorentz group²¹.

Now consider the action of the general linear group $GL(4, \mathbb{C})$ on \mathbb{C}^4 . Such an action does not preserve any “metric” but, nevertheless, can be represented spinorially, if one so

²⁰with $\bar{l} = \bar{L}^{-1}$.

²¹how to represent the discrete symmetries of space-time, the charge conjugation, space and/or time reflections is not discussed in this review, certain aspects of these are however touched upon in [18].

wishes, by

$$\begin{pmatrix} \tilde{\omega}^B \\ \tilde{\pi}_{B'} \end{pmatrix} = \begin{pmatrix} R_A^B & Q^{A'B} \\ U_{AB'} & K_{B'}^{A'} \end{pmatrix} \begin{pmatrix} \omega^A \\ \pi_{A'} \end{pmatrix}, \quad (3.4)$$

where there are no restrictions on the spinor-tensors R , Q , U and K . Of course, the splitting of \mathbb{C}^4 into the two spinor spaces $S \oplus \bar{S}^*$ is not invariant with the non-diagonal blocks present in the $GL(4, \mathbb{C})$ transformations. Complex vectors in \mathbb{C}^4 , although (non-invariantly) represented by a pair of spinors should not be confused with Dirac bispinors: they are **not Dirac bispinors** anymore. Transformations such as in (3.4) are just spinorial representations of the general complex linear group $GL(4, \mathbb{C})$ acting on \mathbb{C}^4 . The two spinors just represent a complex vector Z^α in \mathbb{C}^4 . The same abstract complex vector Z^α in \mathbb{C}^4 can be represented by any pair of spinors related to each other by a transformation such as in (3.4). In this sense, every complex vector Z^α in \mathbb{C}^4 has two levels of representation: first, in terms of abstract spinors and, second in terms of spinor coordinates with respect to a spin frame. It is important to fully understand this point concerning the two levels of representation of an arbitrary complex vector in \mathbb{C}^4 . To a mathematician, such a strange representation of vectors in \mathbb{C}^4 may seem artificial but when the $GL(4, \mathbb{C})$ gets restricted to its subgroup $SU(2, 2)$, the spinorially defined blocks in the $GL(4, \mathbb{C})$ matrix in (3.4) will acquire a physical meaning, as we shall see below in (3.5) and (3.12). The complex vectors in \mathbb{C}^4 become geometrical objects with respect to the “metric” (pseudo-hermitian form) preserved by $SU(2, 2)$.

We proceed to a presentation of $SU(2, 2)$ matrix transformations that constitutes a linear representation of the (four to one covering of the) conformal symmetry group $C(1, 3)$ of the (compactified) affine Minkowski space²² M . In other words the conformal symmetry group of the (compactified) affine Minkowski space appears as a four to one homomorphic image of $SU(2, 2)$. This homomorphism has been analysed at many occasions by Roger Penrose and others, see especially [18, 19] and references therein.

The spinor representation of $SU(2, 2)$ transformations is essential for the blockwise identification of its physically important subgroups such as (the covering of the identity connected element of) the Lorentz subgroup, subgroup of translations (the composition of these two forming the important (covering of the identity connected element of) the Poincaré subgroup), the subgroup of the dilations and the subgroup of the special conformal transformations. All these subgroups live inside $SU(2, 2)$ and act on its complex geometrical vectors Z^α in \mathbb{C}^4 . Using the spinor representation, we deduce that $SU(2, 2)$ matrices t_α^β are of the following form

$$t_\alpha^\beta := \begin{pmatrix} \delta_D^B & 0 \\ iC_{B'D} & \bar{\delta}_{B'}^{D'} \end{pmatrix} \begin{pmatrix} L_E^D & 0 \\ 0 & \bar{l}_{D'}^{E'} \end{pmatrix} \begin{pmatrix} d \delta_M^E & 0 \\ 0 & \frac{1}{d} \bar{\delta}_{E'}^{M'} \end{pmatrix} \begin{pmatrix} \delta_A^M & -iT^{A'M} \\ 0 & \bar{\delta}_{M'}^{A'} \end{pmatrix}, \quad (3.5)$$

where in (3.5), the first matrix component from the right corresponds to translations ($T^a = T^{A'A}$) in the *affine Minkowski space*, the second matrix component from the right corresponds to scale changes (d^2 , where d is a non-zero real or purely imaginary number) of all Lorentz vectors and all position four-vectors in the *affine Minkowski space*, the third

²²on the (compactification of the) affine M the action of $C(1, 3)$ is non-linear.

matrix component from the right represents (spinorial) Lorentz transformations ($L_B^A, \bar{l}_{A'}^{B'}$) and finally, the fourth matrix component from the right corresponds to special conformal transformations ($C_{A'B}$) of the *affine Minkowski space*. Allowing all permutation of the four component matrices in (3.5) defines the entire set of $SU(2, 2)$ matrices.

A partial proof of this claim now follows.

The first easy thing to note is that the transformations resulting from compositions given by (3.5) (or given by any permutation of the four transformations in (3.5) building up the total transformation), do *not preserve* the metric in (3.2), in general. This happens because there are non-diagonal spinor blocks in the representation of total matrix t_α^β in (3.5). The decomposition of T into a direct sum of two spinor spaces is, therefore, not invariant. What happens is that the two spinor spaces mix with each other if translations and special conformal transformations are performed. Consequently, as mentioned before, the geometrical vectors (non-projective twistors) of T (on which the total transformation t_α^β of the type such as in (3.5) act) should not be confused with the Dirac bispinors. The second easy thing to note is that the determinants of all total transformations t_α^β , composed the four component matrices of the type as in (3.5), are always equal to 1. This happens because the determinat of any of its four matrix components is equal to 1, as follows from a simple inspection of the representation of the component matrices in (3.5). We will now show that the set of total transformations of the type (3.5) is also unitary and preserves the pseudo-hermitian form g explicitly represented by²³

$$g_{\alpha\beta} := \begin{pmatrix} 0 & \bar{\delta}_{A'}^{B'} \\ \delta_B^A & 0 \end{pmatrix}, \quad (3.6)$$

and its inverse is given by

$$g^{\alpha\beta} := \begin{pmatrix} 0 & \delta_A^B \\ \bar{\delta}_{B'}^{A'} & 0 \end{pmatrix}. \quad (3.7)$$

We have just to show that for all total transformations²⁴ t_α^β of the type (3.5), the following properties are valid:

$$\bar{t}_\delta^\beta t_\gamma^\alpha g_{\alpha\beta} = g_{\delta\gamma}, \quad (3.8)$$

$$(\bar{t}^T)_\beta^\alpha = (t^{-1})_\beta^\alpha, \quad (3.9)$$

where the superscript index T indicates the transposition, the bar (as always in this paper) over a letter indicates complex conjugation while the superscript -1 indicates the inverse.

To verify identities in (3.8) and (3.9), it is enough do it separately for each component of the total transformation in (3.5). For example, if we take the translation component of a

²³note that the signature of the matrices in (3.6) and (3.7) is $++--$, easily verified by diagonalisation. The spinor representation of vectors in \mathbb{C}^4 requires the representation of the $SU(2, 2)$ invariant pseudo-hermitian form (the “metric”) to appear in this non-diagonal spinorial disguise.

²⁴any permutation of the the four matrix components in (3.5) gives such a total transformation.

transformation of the form (3.5), we verify (3.8) by noting that

$$\begin{pmatrix} \bar{\delta}_{B'}^{A'} & 0 \\ iT^{A'B} & \delta_A^B \end{pmatrix} \begin{pmatrix} 0 & \bar{\delta}_{A'}^{K'} \\ \delta_K^A & 0 \end{pmatrix} \begin{pmatrix} \delta_L^K & -iT_{K'}^L \\ 0 & \bar{\delta}_{K'}^{L'} \end{pmatrix} = \begin{pmatrix} 0 & \bar{\delta}_{B'}^{L'} \\ \delta_L^B & 0 \end{pmatrix}. \quad (3.10)$$

Identity (3.9) for the translation component follows by a similar argument:

$$\begin{aligned} & \overline{\left(\begin{pmatrix} 0 & \bar{\delta}_{A'}^{K'} \\ \delta_K^A & 0 \end{pmatrix} \begin{pmatrix} \delta_M^K & -iT^{M'K} \\ 0 & \bar{\delta}_{K'}^{M'} \end{pmatrix} \begin{pmatrix} 0 & \delta_L^M \\ \bar{\delta}_{M'}^{L'} & 0 \end{pmatrix} \right)^T} \\ &= \overline{\begin{pmatrix} \bar{\delta}_{A'}^{L'} & 0 \\ -iT^{L'A} & \delta_A^L \end{pmatrix}}^T = \begin{pmatrix} \delta_A^L & iT^{A'L} \\ 0 & \bar{\delta}_{L'}^{A'} \end{pmatrix}. \end{aligned} \quad (3.11)$$

Here, the last matrix represents the inverse of the translation, as asserted. The remaining verifications are left as a spinor algebra exercise to the reader. This completes our partial proof of the fact that the matrices arising as the result of multiplication of the four components (taken in any order) displayed in (3.5) constitute the matrix group $SU(2, 2)$.

For future reference, an explicit formula transforming the spinor representatives of a twistor, and arising as a result of the composition of the four symmetry transformations in (3.5), is displayed here

$$\tilde{Z}^\alpha = t_\beta^\alpha Z^\beta,$$

or, equivalently,

$$\begin{pmatrix} \tilde{\omega}^B \\ \tilde{\pi}_{B'} \end{pmatrix} = \begin{pmatrix} dL_A^B & -id T^{A'D} L_D^B \\ id C_{B'D} L_A^D & d C_{B'D} T^{A'F} L_F^D + \frac{1}{d} \bar{l}_{B'}^{A'} \end{pmatrix} \begin{pmatrix} \omega^A \\ \pi_{A'} \end{pmatrix}. \quad (3.12)$$

Consider now two contravariant twistors given by their spinor representatives as follows

$$Z^\alpha := \begin{pmatrix} \omega^A \\ \pi_{A'} \end{pmatrix}, \quad W^\alpha := \begin{pmatrix} \lambda^A \\ \eta_{A'} \end{pmatrix}. \quad (3.13)$$

Their “covariant” versions read

$$\bar{Z}_\alpha := g_{\alpha\beta} \overline{Z^\beta} = (\bar{\pi}_A, \quad \bar{\omega}^{A'}), \quad \bar{W}_\alpha := g_{\alpha\beta} \overline{W^\beta} = (\bar{\eta}_A, \quad \bar{\lambda}^{A'}). \quad (3.14)$$

Using the spinor representation of the conformally $(SU(2, 2))$ invariant complex valued “scalar” product (known by mathematicians as a pseudo-hermitian form) in the twistor space, we can write it as follows

$$\rho := Z^\alpha \bar{W}_\alpha = g_{\alpha\beta} Z^\alpha \overline{W^\beta} = \omega^A \bar{\eta}_A + \bar{\lambda}^{A'} \pi_{A'}. \quad (3.15)$$

Note that the “length” (pseudo-hermitian norm) of any twistor is always a real number, either positive or negative or null.

As well-known to mathematicians, the imaginary part of the pseudo-hermitian form preserved by the action of the matrix group $SU(2, 2)$ defines on $T \simeq \mathbb{C}^4$ a conformally

invariant symplectic structure²⁵ which may be expressed in terms of global canonical and $SU(2, 2)$ invariant Poisson bracket commutation relations as

$$\{\bar{Z}_\beta, Z^\alpha\} = i\delta_\beta^\alpha, \quad (3.16)$$

with all the remaining Poisson bracket commutation relations being equal to zero. In terms of the twistor's spinor representatives it reads

$$\{\bar{\pi}_B, \omega^A\} = i\delta_B^A. \quad (3.17)$$

The set of all twistors $\{Z, W, \dots\}$ in T modulo multiplication by a non-zero complex number defines its projective counterpart $\mathbb{CP}(3)$. Depending on the sign of the “length” (pseudo-hermitian norm) of a twistor, $\mathbb{CP}(3)$ is divided into three parts. The first part of $\mathbb{CP}(3)$ corresponds to twistors of positive norm, the second part to twistors of negative norm and the third to twistors with zero norm.

Up to now we have been quite silent about the physical interpretation of the elements (complex vectors) in the twistor space $T \simeq \mathbb{C}^4$ and their projective counterparts (complex lines through the origin in T) in the projective twistor space $\mathbb{CP}(3)$, except for the fact that they are natural carriers of the conformal (i.e. $SU(2, 2)$) symmetry. As shown by Penrose, each twistor may be interpreted in the Minkowski space both geometrically and dynamically.

Geometrically, any element in $\mathbb{CP}(3)$ can be identified with a shear-free and, in general, twisted congruence (Robinson congruence) of null-lines filling up the entire affine Minkowski space or, alternatively, with the so called α -plane in a complexified Minkowski space. In this paper, we will not discuss in detail the Minkowski geometrical interpretation of twistors, as we expect to obtain the Minkowski space time events (i.e. affine *position* four-vectors) themselves as twistor constructions. We wish thus to follow the same pattern as in the previous section where Lorentz four-vectors and, thereby, the entire Minkowski vector space appeared as a subspace of the spinor-tensor algebra (namely, as a set of all hermitian spinor-tensors of mixed second rank). More information on interpretation of twistors as geometrical objects in (also curved) space-time can be found in [13, 16, 19, 20]. In this context, the holomorphic aspects of complex valued functions of twistors are very important, allowing to perform geometrical constructions of general solutions to well-known equations in physics, not to omit the famous non-linear (and anti-selfdual) graviton construction. However, these fascinating topics, as said above, will not be discussed in this paper.

The dynamical/kinematical description of a twistor is most easy to grasp by using the spinor representation of twistors and by restricting (at least at the first instance) the $SU(2, 2)$ matrix group action to its “Poincaré” subgroup by putting $C_{A'B} = 0$ and $d = 1$ in (3.5). We allow also the remaining two matrix components of the matrix t_β^α in (3.5) to appear in the reverse order. This defines completely (the covering group of the identity connected part of) the Poincaré group. A very similar representation of the Poincaré group was also considered by Bogoliubov, Todorov and Lagunov in their monograph [6].

²⁵simply meaning that conformal transformations (i.e. $SU(2, 2)$ transformations) form a closed subgroup of all canonical transformations preserving such a symplectic structure.

Using the spinor representatives of a twistor, it is quite easy now to extract the dynamical/kinematical quantities. A twistor defines the total four-angular momentum (including its orbital part and therefore represented by a *translation dependent* antisymmetric Lorentz tensor of second rank) of a massless object, its total linear four-momentum (a true Lorentz null four-vector) and also (which is a genuine new feature) its (classical limit of the) helicity²⁶:

$$P_a := \pi_{A'} \bar{\pi}_A \quad \text{four-linear momentum,} \quad (3.18)$$

$$M_{ab} := i\bar{\omega}_{(A'} \pi_{B')} \epsilon_{AB} - i\omega_{(A} \bar{\pi}_{B')} \epsilon_{A'B'} \quad \text{four-angular momentum,} \quad (3.19)$$

where we used spinor representation of Lorentz tensors in accordance with the previous section. Note that with respect to translations represented by the first component from the right in (3.5), the angular four momentum has correct transformation properties²⁷. The (classical) helicity arises by extracting the translation invariant spin contribution to the total angular four-momentum in (3.19). This is achieved in the usual way by forming the so called Pauli-Lubański spin four-vector from (3.18)-(3.19). By performing some elementary spinor tensor algebra we obtain²⁸

$$S^a := \frac{1}{2} \epsilon^{abcd} M_{bc} P_d = s P^a = s \pi^{A'} \bar{\pi}^A \quad \text{where} \quad s := \frac{1}{2} (Z^\alpha \bar{Z}_\alpha). \quad (3.20)$$

A quite remarkable fact is that the real valued conformally invariant function s in (3.20), i.e. half of the $SU(2,2)$ norm of Z , defines (the classical limit) of the helicity operator of a massless object (particle).

In addition, it turns out also that the canonical conformally invariant Poisson bracket relations in (3.17) imply that the real valued, Poincaré covariant functions on T , representing physical variables of a massless particle as spinorially defined in (3.18)-(3.19), fulfill the Poisson bracket relations of the Poincaré algebra²⁹:

$$\{P_a, P_b\} = 0, \quad (3.21)$$

$$\{M_{ab}, P_c\} = -P_a \eta_{bc} + P_b \eta_{ac}, \quad (3.22)$$

$$\{M_{ab}, M_{cd}\} = M_{ac} \eta_{bd} + M_{bd} \eta_{ac} - M_{ad} \eta_{bc} - M_{bc} \eta_{ad}. \quad (3.23)$$

It is obvious that the *non-invariant* translation spinor part, “omega”, of a twistor has to do with the Minkowski position four-vector (take a look at the first matrix component from the right in (3.5) acting on twistors), which is the important element in the usual definition of the orbital part of the total angular four-momentum, here defined twistorially

²⁶note that the two dynamical quantities in (3.18)-(3.19) are only Poincaré invariant (covariant) while the helicity is also a conformally invariant scalar, as will be shown below in (3.20).

²⁷exercise: show this using the Fierz identity in (2.22).

²⁸try to fill in the details of this calculation yourself. Otherwise look it up in [19].

²⁹a tedious spinor algebra calculation can prove that, you want to try? Do you have another method to prove it?

in (3.19). To figure out where the Minkowski four-position variables may be hidden inside the “omega” part of a twistor we represent it spinorially as follows (by a spinor-tensor contraction)³⁰

$$\omega^A := iz^{AA'}\pi_{A'}. \quad (3.24)$$

In (3.24) $z^{AA'}$ is a mixed spinor tensor of second rank (not necessarily hermitian) that is equivalent with a complexified Lorentz four-vector whose real part (i.e. hermitian part of $z^{AA'}$) transforms under the action of four translations (the first matrix component from the right in (3.5)) as a position four-vector in the *affine* Minkowski space. Therefore, the real part of z is a good candidate to represent physical space-time events that massless twistor objects trace out in the affine Minkowski space. However, $z^{AA'}$ is not uniquely determined by the “omega” and “pi” parts of twistors because any complexified Lorentz four-vector (with its real part defining a set of events in the affine Minkowski space) of the form

$$\check{z}^{AA'} = z^{AA'} + \lambda\alpha^A\pi^{A'}, \quad (3.25)$$

where λ is any complex valued parameter and where α^A is an arbitrarily chosen spinor (zero spinor excluded), would do.

The equation in (3.25) defines what is nowadays called an α -plane in the complexified Minkowski space. Each “contravariant” (with respect to the $SU(2,2)$ “metric”) twistor Z defines such a plane. The corresponding “covariant” (with respect to the $SU(2,2)$ “metric”) twistor \bar{Z} defines another such a plane that is called β -plane.

If the norm of a twistor vanishes (helicity in (3.20) equals zero), then complexified position spinor-tensors (complexified position four-vectors) $z^{AA'}$ and $\check{z}^{AA'}$ become hermitian (i.e. complexified four-vectors become real) and the two complex planes merge into a real line defining a single null-line in the real affine Minkowski space:

$$\check{x}^{AA'} = x^{AA'} + \lambda\bar{\pi}^A\pi^{A'}, \quad (3.26)$$

where λ is now an arbitrary *real* parameter. This is the famous correspondence between zero norm twistors and null-lines in the Minkowski space [9, 12, 14, 15, 18, 19, 20, 21]. α -plane can also be interpreted as the so called right handed twisting Robinson congruence in the real Minkowski space while the corresponding β -plane may then be interpreted as almost the same twisting Robinson congruence but with the twist reversed. Besides, the handedness of the twist is determined by the sign of the twistor norm (remember this is just a real number, positive, null or negative). As we mentioned above, we will not be very informative about the space-time interpretations of twistors. Nevertheless, the above tiny piece of information (without any proofs) may add some understanding to the ideas involved in the twistor formalism.

There exists a large number of articles about the massless particles and massless fields described in terms of twistors, as we briefly referred. The interested reader is advised to

³⁰the imaginary number i is inserted in order to harmonize with the the adopted notational conventions as e.g. the representation of the $SU(2,2)$ explicitly given in (3.5).

go to the original papers [9, 12, 14, 15, 18, 19, 20, 21] and references therein. We will, however, not dwell on this massless case much more but turn to the case of the simplest (classical limit of the) massive object that can be constructed in terms of twistors. The phase space nature (symplectic structure) of the twistor space will be our main tool.

For that reason consider now a direct product of two-twistor spaces $T \times T$ with its diagonal deleted³¹. The resulting space will be denoted by $T\Delta T$. For each element (Z, W) in $T\Delta T$ we require that $Z^\alpha \neq l W^\alpha$, where l is any complex number. This can be formulated in another way: we require that the two twistors in the pair do not define the same point in $\mathbb{CP}(3)$.

By linearity (see (3.16)), the phase space structure on $T\Delta T$ is given by the following conformally invariant global canonical Poisson bracket commutation relations

$$\{\bar{Z}_\beta, Z^\alpha\} = i\delta_\beta^\alpha, \quad \{\bar{W}_\beta, W^\alpha\} = i\delta_\beta^\alpha \quad (3.27)$$

with all the remaining Poisson bracket commutation relations being equal to zero. In terms of the spinor representatives, the two twistors are given by

$$Z^\alpha = (\omega^A, \pi_{A'}) \quad \text{and} \quad W^\alpha = (\lambda^A, \eta_{A'}), \quad (3.28)$$

so that the conformally invariant canonical global Poisson bracket commutation relations in (3.27) may be written (see (3.17)) as

$$\{\bar{\pi}_B, \omega^A\} = i\delta_B^A, \quad \{\bar{\eta}_B, \lambda^A\} = i\delta_B^A \quad (3.29)$$

with all the remaining Poisson bracket commutation relations being equal to zero.

Two each twistor in the pair there corresponds an α -plane (and also a β -plane) in the complexified Minkowski space. Therefore, the intersection of the two non-coinciding, by definition because the diagonal was excluded, α -planes meet in a single complexified z position Lorentz four-vector. Therefore, an arbitrary point in $T\Delta T$ defines such a z explicitly. We just need to solve the following set of equations

$$\omega^A = iz^{AA'}\pi_{A'}, \quad \lambda^A = iz^{AA'}\eta_{A'}. \quad (3.30)$$

The solution of the equations in (3.30) reads (see [19, 20])

$$z^{AA'} = \frac{i}{f}(\omega^A\eta^{A'} - \lambda^A\pi^{A'}), \quad (3.31)$$

where

$$f := \pi^{A'}\eta_{A'} \neq 0, \quad (3.32)$$

³¹in section two, in order to define a non-null Lorentz four-vector in terms of spinors we needed (at least) two of them. The two were not allowed to be proportional to each other. In the same way, to define a massive object in terms of twistors we need (at least) two of them. They must not be proportional to each other.

the last inequality being true just because we are in $T\Delta T$.

It is tempting to define the Lorentz position four-vector as the hermitian part of the solution (i.e. the real part of the corresponding complexified position four-vector) in (3.31). This would mean that we take the complex intersection point of the two α planes defined by the two “contravariant” twistors Z, W and the complex intersection of the corresponding two β planes defined by the corresponding “covariant” twistors \bar{Z}, \bar{W} and take the mean value of them which is then hermitian, i.e. defines a real position four-vector. Explicitly, such a Poincaré covariant hermitian (i.e. real position four-vector) spinor-tensor of mixed second rank q reads

$$q^a = q^{AA'} := \frac{1}{2}(z^{AA'} + \bar{z}^{A'A}), \quad (3.33)$$

where, of course,

$$\bar{z}^{A'A} = \frac{-i}{f}(\bar{\omega}^{A'}\bar{\eta}^A - \bar{\lambda}^{A'}\bar{\pi}^A). \quad (3.34)$$

Before we discuss further the relevance of the definition of the position four-vector q in (3.33) we identify the angular four-momentum and linear four-momentum of the two-twistor object in $T\Delta T$. This must be done in such a way that the symplectic structure on $T\Delta T$ inherited from the symplectic structure on T (being the imaginary part of the pseudo-hermitian metric on T , preserved by $SU(2,2)$) still implies that they fulfill the commutation relations of the Poincaré Poisson bracket algebra as in (3.21) - (3.23). The task is easy because, by linearity, we can simply add variables describing the two (mutually Poisson commuting) massless parts (see (3.18)-(3.19)) and obtain

$$P_a = P_{A'A} := \pi_{A'}\bar{\pi}_A + \eta_{A'}\bar{\eta}_A, \quad (3.35)$$

$$\begin{aligned} M_{ab} &= M_{A'AB'B} \\ &:= i(\bar{\omega}_{(A'}\pi_{B')}\epsilon_{AB} - \omega_{(A}\bar{\pi}_{B')}\epsilon_{A'B'}) + i(\bar{\lambda}_{(A'}\eta_{B')}\epsilon_{AB} - \lambda_{(A}\bar{\eta}_{B')}\epsilon_{A'B'}). \end{aligned} \quad (3.36)$$

The angular and linear four-momenta defined in (3.35) - (3.36) will automatically fulfil the commutation relations of the Poincaré Poisson bracket algebra. Using our definition of the position four-vector in (3.33), it turns out that the angular four momentum in (3.36) can be written as³²

$$M^{ab} = P^a q^b - P^b q^a + S^{ab}, \quad (3.37)$$

$$S^{ab} := \frac{1}{(P^k P_k)} \epsilon^{abcd} P_c S_d,$$

and where the Pauli-Lubański four-vector S^a is defined in the usual way,

$$S^a := \frac{1}{2} \epsilon^{abcd} M_{bc} P_d. \quad (3.38)$$

³²to verify this requires a tedious spinor algebra calculation.

The fact displayed in (3.37) would support our choice of the definition of the position four-vector variable in the phase space $T\Delta T$ according to (3.33). However, there is one great disadvantage because the so defined position four-vector is non-commuting; instead, the following Poisson bracket commutation relations may be derived [4, 5, 28]:

$$\{q^a, q^b\} = -S^{ab}. \quad (3.39)$$

The above non-commuting feature of the four-position variable, in the case when the intrinsic angular momentum part is assumed to be effectively defined by S^a representing only three variables (S^a in (3.38) automatically fulfils $S^a P_a = 0$), seems to be an entirely generic feature in the geometry of the so called relativistic extended phase spaces as discovered by professor Stanisław Zakrzewski [28, 29, 30]. His discoveries (being an extension and generalisation of certain mathematical results obtained by J.M. Souriau [24]) were made without any use of twistors and classify all Poincaré invariant extended phase spaces. It has been proved in [4, 5] that one of the cases considered by him is a “subset” of the two-twistor construction as presented in this report.

Our conjecture is that with three or more twistors the Zakrzewski’s general feature would reappear in the twistor formalism and perhaps could shed some light on the physical relevance of Zakrzewski’s and Souriau’s mathematical achievements.

The four-position q , in (3.33), will be called the centre of mass of the massive spinning and charged system defined by a point in $T\Delta T$. It has been discovered in [4, 5] that a redefinition of the position four-vector may be found in such way that the commutation of the four-position variables will be restored. Defining a new four-position:

$$x^a = x^{AA'} := q^{AA'} + \Delta x^{AA'}, \quad (3.40)$$

where

$$\Delta x^{AA'} := \frac{i}{2f\bar{f}}(\rho\bar{\pi}^A\eta^{A'} - \bar{\rho}\pi^{A'}\bar{\eta}^A), \quad (3.41)$$

we obtain [4, 5] that³³

$$\{x^a, x^b\} = 0. \quad (3.42)$$

The new commuting four-position x (in (3.40)) shifted by Δx (in (3.41)) away from the (non-commuting) centre of mass q (in (3.33)) will be called the centre of charge of the system. The total angular four-momentum now splits in a different way compared with

³³all these statements may be checked by hand using the spinor representatives of the twistors and the spinor algebra rules but this is tedious and not very amusing. The ambitious reader is encouraged to do the necessary calculations and find possible minor omissions or/and sign errors if any. There is also the so called Penrose’s blob notation that simplifies such calculations a lot but first you have to be able to master the blob notation :).

the splitting in (3.37) because the orbital momentum is defined with respect to the centre of charge instead. We obtain [4, 5]:

$$M^{ab} = P^a x^b - P^b x^a + \Sigma^{ab}, \quad (3.43)$$

where

$$\Sigma_{ab} := [\sigma_{(A'} \eta_{B')} \epsilon_{AB} + \bar{\sigma}_{(A} \bar{\eta}_{B)} \bar{\epsilon}_{A'B'}], \quad (3.44)$$

$$\sigma_{A'} := \frac{i}{f} (k \pi_{A'} + \rho \eta_{A'}), \quad (3.45)$$

$$k := s_1 - s_2, \quad (3.46)$$

$$s_1 := \frac{1}{2} (Z^\alpha \bar{Z}_\alpha), \quad s_2 := \frac{1}{2} (W^\alpha \bar{W}_\alpha), \quad \rho := (Z^\alpha \bar{W}_\alpha). \quad (3.47)$$

It is relatively easy to realize that the translation invariant antisymmetric spin four-tensor Σ is composed of two parts: one part defining rotation of the centre of mass around the centre of charge³⁴, and one part defining the intrinsic rotation of the centre of charge itself. The number of variables it defines is five and not six, as one would expect from a general antisymmetric Lorentz four-tensor of rank two (but even five is contrasting with the three variables defined by the Pauli-Lubański Lorentz four-vector S^a in (3.38)) because of the vanishing of the Poincaré scalar³⁵

$$\Sigma^{ab} \epsilon_{abcd} \Sigma^{cd} = 0, \quad (3.48)$$

which may also be verified by direct spinor algebra calculations. Note also that

$$\Sigma^{ab} \Sigma_{ab} = -2[(\sigma^{A'} \eta_{A'})^2 + (\bar{\sigma}^A \bar{\eta}_A)^2] = 4k^2, \quad (3.49)$$

where, remarkably, $\Sigma^{ab} \Sigma_{ab} = 4k^2$ is not only Poincaré but also conformally invariant scalar function.

The variables x in (3.40) and P in (3.35) define the eight dimensional phase space that can be identified with the cotangent bundle of the real Minkowski space. The implied (by (3.29)) Poisson bracket commutation relations read

$$\{P_a, x^b\} = \delta_a^b, \quad (3.50)$$

as they should. Moreover, Σ commutes with x in (3.40) and with P in (3.35) as shown in detail in [4, 5]. Σ defines a second rank antisymmetric Lorentz four-tensor valued function on a six dimensional Poincaré invariant phase space identified with the cotangent bundle

³⁴this could be the root of the so called “Zitterbewegung”, see our further developments in this review and [7].

³⁵this feature is a consequence of the two-twistor construction, with three or more twistors, perhaps the invariant in (3.48) would be different from zero.

of the real projective spinor space [4], the latter spanned by the Poincaré invariant spinor variable η (of the second twistor W in the pair) modulo its multiplication by a non-zero real number. The Poincaré invariant variable in the cotangent fiber is then the above defined σ spinor (in (3.45)) modulo its multiplication by the inverse of this real number. Σ fulfills the commutation relations of the Lorentz Poisson bracket algebra,

$$\{\Sigma_{ab}, \Sigma_{cd}\} = \Sigma_{ac}\eta_{bd} + \Sigma_{bd}\eta_{ac} - \Sigma_{ad}\eta_{bc} - \Sigma_{bc}\eta_{ad}. \quad (3.51)$$

as it should do. The commutations relations in (3.51) can be computed by using the canonical commutation relations defining the Poincaré invariant symplectic structure on the cotangent bundle of the real spinor space (spinor modulo its multiplication by a real number) or more simply using (3.27) alternatively (3.29).

Finally, the conformally invariant scalar function

$$e := 2s_1, \quad (3.52)$$

(we call it the classical limit of the electric charge quantum operator³⁶ with s_1 defined in (3.47)) and $\arg f$ (where f , the Poincaré invariant scalar function, was defined in (3.32)), commute with x in (3.40) and P in (3.35) and with all the variables of the cotangent bundle of the real spinor space, as defined above, so that they form an independent cotangent bundle over a circle. The implied commutation relations read

$$\{e, \arg f\} = 1. \quad (3.53)$$

We have in this way (details in [4, 5]) constructed a Poincaré invariant decomposition of the two twistor $T\Delta T$ phase space (16 dimensions) into three independent parts: a cotangent bundle of the Minkowski space (8 dimensions), a cotangent bundle of a real projective spinor space (6 dimensions) and a cotangent bundle over a circle (2 dimensions). The first two parts forming a 14 dimensional extended phase space, i.e. the cotangent bundle of the Minkowski space (8 dimensions) and the cotangent bundle of a real projective spinor space (6 dimensions) constitute (as already mentioned above) a special case of the very general geometrical mathematical construction obtained, without any use of twistors, by professor Stanisław Zakrzewski in [28, 29, 30].

There is a peculiar discrete symmetry in our construction. The entire construction works equally well if we change the sign in front of the shift Δx^a in (3.40) and interchange simultaneously the role of the two spinors $\pi \rightarrow \eta$, $\eta \rightarrow \pi$. The charge variable will then be defined by the norm of the second twistor (that is, $e := 2s_2$) instead. Further analysis of this peculiarity and its possible interpretation in particle physics could perhaps be of value.

The decomposition of the phase space $T\Delta T$ as constructed above³⁷ is Poincaré invariant. However, it is not invariant with respect to the special conformal transformations, i.e. it is

³⁶remarkably, it appears as a conformal scalar, twice the helicity of the first massless constituent, i.e. as the norm of the first twistor in the pair. Here our identification of the charge differs from that usually assumed by people in the Penrose's group, see, for example, [9, 21, 25, 27]. They assume $s_1 + s_2$ to be the electric charge but such a choice would destroy our symplectic Poincaré invariant decomposition of $T\Delta T$.

³⁷in much greater detail described in [4, 5].

not invariant with respect to the action of the fourth component matrix from the right in (3.5). Only the value of the charge e , the value of $\Sigma^{ab}\Sigma_{ab} = 4k^2$ and, in fact, the Lorentz norm of the Pauli-Lubański spin four-vector divided by the norm of the (canonical) linear momentum four-vector $P^a P_a = 2f\bar{f}$:

$$w^2 := \frac{-S_a S^a}{P^b P_b}, \quad (3.54)$$

are, by definition, also preserved by the action of the special conformal transformations (and dilations). The proof of the conformal invariance of w^2 in (3.54) follows from a straight forward spinor algebra calculations. We get that [21, 2]

$$w^2 = k^2 + \rho\bar{\rho}, \quad (3.55)$$

where k and ρ were previously defined in (3.46) and in (3.15). They are, by definition, conformal ($SU(2, 2)$) scalar invariants.

It is therefore interesting to see how the action of the fourth component matrix from the right in (3.5) affect the Poincaré covariant/invariant variables such as q , x , Δx , P , S and Σ defined in (3.33), (3.40), (3.41), (3.35), (3.38), (3.44).

For that reason we will determine how the conformal transformations affect z (i.e. the intersection of the two α planes defined by the two twistors) and how they affect the Poincaré invariant/covariant³⁸ spinors π , η . Some lengthy spinor algebra calculations give

$$\tilde{z}^a = \frac{z^a - \frac{1}{2}C^a(z_k z^k)}{1 - C_b z^b + \frac{1}{4}(C_n C^n)(z_m z^m)}, \quad (3.56)$$

where we used the identifications: $z^a = z^{AA'}$ (see (3.31)) and $C_a = C_{A'A}$ (see (3.5)). Note that C is a real Lorentz four-vector while z , in general, is not, with its real part (in the affine Minkowski space) representing the (non-commuting) position four-vector q of the centre of mass of the two-twistor particle (spinning and electrically charged object)³⁹.

The special conformal transformation of the intersection of the corresponding two β planes, defined by the two twistors, is obtained by a simple complex conjugation of (3.56). If the norms of the two twistors s_1 , s_2 and the “scalar” product ρ all vanish, i.e. if $s_1 = s_2 = \rho = 0$, then the two-twistor particle is still massive but uncharged and non-spinning. Its z Lorentz four-vector becomes real [19], i.e. the two α and the two β planes merge into two distinct null-lines with a real intersection defining the Lorentz four-position $z = q = x$ transforming under special conformal transformations in the well-known traditional way.

For an arbitrary element in $T\Delta T$, i.e. for a general relativistic massive, spinning and charged object, however, our definition of the Minkowski position four-vectors goes beyond

³⁸we use the word invariant and covariant interchangeably; if by a transformation we mean a change of the coordinate system then the geometrical objects are invariant, only their coordinates are changing; on the other hand, if the coordinate system is kept unchanged then every geometrical objects is covariantly transformed into another one; the two meanings merge into one when it comes to geometrical scalars.

³⁹it should perhaps be mentioned that the coordinate function of the complexified position four-vector in (3.31) do, in fact, commute, which is easy to see from (3.29).

the usual definition because under the action of special conformal transformations (the fourth matrix component in the righthandside of (3.5)), the real Lorentz position four-vector q mixes with the spin, electric charge and the linear momentum variables of the two-twistor particle in the way indicated by (3.56). To see that one should recall that the imaginary part of z contains variables describing spin and charge and is given by [2, 4, 5, 9, 27]:

$$Y^a := \frac{1}{2i}(z^a - \bar{z}^a) = \frac{1}{2ff}[\rho\bar{w}^a + \bar{\rho}w^a - 2s_1(P_2)^a - 2s_2(P_1)^a], \quad (3.57)$$

where

$$(P_1)_a := \pi_{A'}\bar{\pi}_A, \quad (P_2)_a := \eta_{A'}\bar{\eta}_A, \quad w_a := \pi_{A'}\bar{\eta}_A. \quad (3.58)$$

The action of a special conformal transformation on the Poincaré invariant/covariant spinors π and η can be expressed explicitly as

$$\tilde{\pi}_{A'} = \pi_{A'} + iC_{A'A}\omega^A = [1 - \frac{1}{2}C_a z^a] \pi_{A'} + \frac{1}{2}\bar{\epsilon}^{AB}(C_{AA'}z_{BB'} - C_{BB'}z_{AA'}) \pi^{B'}, \quad (3.59)$$

$$\tilde{\eta}_{A'} = \pi_{A'} + iC_{A'A}\omega^A = [1 - \frac{1}{2}C_a z^a] \eta_{A'} + \frac{1}{2}\bar{\epsilon}^{AB}(C_{AA'}z_{BB'} - C_{BB'}z_{AA'}) \eta^{B'}. \quad (3.60)$$

These formulae follow directly from (3.5).

The transformation properties of the two spinors⁴⁰ π , η under the action of the special conformal transformations ((3.59), (3.60)) are new features arising as a consequence of the suggested twistor representation of (the classical limit of) the relativistic physical variables. We think that more conclusions could be drawn from this observation. However, we will not investigate the issue any further in this review.

We proceed to an application of the formalism developed above. In the next two sections, the Poincaré invariant dynamics of a relativistic massive, charged and spinning two-twistor particle will be formulated on $T\Delta T$.

4 MINIMALLY COUPLED SECOND ORDER DIRAC OPERATOR AND ITS CLASICAL LIMIT ON THE TWO-TWISTOR SPACE.

First we present some well-known facts (see e.g.[23] and [26] pp.88-89 and Appendix A) about the minimally coupled Dirac equation using the language of spinors developed in the second section of the paper. The electro-magnetic Lorentz four-vector potential expressed as an hermitian second rank mixed tensor field reads

$$A_c(x^d) = A_{CC'}(x^{DD'}), \quad (4.1)$$

⁴⁰in (3.35) they were used to define the linear four-momentum, i.e. it represented an element in the “cotangent” fiber over the four-position (in the base manifold) of the massive two-twistor relativistic object. The special conformal transformations destroy our Poincaré invariant symplectic decomposition of $T\Delta T$ mapping it onto a new one.

where $x^{DD'}$ is a translation dependent hermitian mixed second rank spinor-tensor representing position four-vectors, i.e. labeling points in the Minkowski affine space. At this stage we do not need to associate them with twistors. The components of $x^{DD'}$ are then mutually commuting by definition.

In relativistic quantum mechanics the (mechanical and not canonical) linear four-momentum operator of a charged massive quantum particle in an external electro-magnetic field is defined by⁴¹

$$\hat{p}_{BA'} := i \frac{\partial}{\partial x^{BA'}} - e A_{BA'}(x^{CD'}), \quad (4.2)$$

where e represents the numerical value of the particle's electric charge. The canonical differential operator defined by

$$i\partial_{KM'} := i \frac{\partial}{\partial x^{KM'}}, \quad (4.3)$$

represents particle's canonical linear four-momentum operator (being essentially the same as the infinitesimal generator of space-time four-translations) in the absence of interactions. The definition in (4.2), on the other hand, represents the *minimal coupling* of the charged particle⁴² to an external electro-magnetic field defined by its potential four-vector $A_a(x^b)$.

A Dirac bispinor field⁴³ over the Minkowski space

$$\Psi^A := \Psi^A(x^{CD'}), \quad \Phi_{A'} := \Phi_{A'}(x^{CD'}) \quad (4.4)$$

is said to obey the Dirac equation if

$$\hat{p}^{AB'} \Phi_{B'} = \frac{m}{\sqrt{2}} \Psi^A, \quad \hat{p}_{BA'} \Psi^B = \frac{m}{\sqrt{2}} \Phi_{A'}. \quad (4.5)$$

Here (4.5) is interpreted as a classical (i.e. non-quantum) formula and if the bispinors had constant values, then one could say, with reference to the discussion in section two of this review, that the Dirac equation simply states that the linear four-momentum is a sum of two null vectors constructed from two distinct spinors Φ and Ψ with their $\text{SL}(2, \mathbb{C})$ “scalar” product normalised to a real value⁴⁴ $\frac{m}{\sqrt{2}}$ where m is identified with the mass of the particle (this interpretation would exclude the Majorana-Weyl bispinor because in this case the $\text{SL}(2, \mathbb{C})$ “scalar” product is equal to zero). The Dirac equation can be viewed also as an eigenvalue problem involving the mass m as the spectral parameter because it can be written in a suggestive way as

$$\begin{pmatrix} 0 & \hat{p}^{AB'} \\ \hat{p}_{BA'} & 0 \end{pmatrix} \begin{pmatrix} \Psi^B \\ \Phi_{B'} \end{pmatrix} = \frac{m}{\sqrt{2}} \begin{pmatrix} \Psi^A \\ \Phi_{A'} \end{pmatrix}. \quad (4.6)$$

⁴¹we choose units so that $c = \hbar = 1$.

⁴²with the value of its charge being equal to e .

⁴³at this first quantized stage the components of the spinors are not anticommuting operator valued distributions but usual complex number fields.

⁴⁴ $\Psi^A \bar{\Phi}_A = \frac{m}{\sqrt{2}}$.

Looked upon in this way (i.e. as in (4.6)), the linear four-momentum operator acts as an infinitesimal four-translation operator on the components of the bispinor fields, simultaneously mixing the two spinor fields with each other. In the so called second order formulation of the Dirac equation the mixing can be avoided and the Dirac equation gets a clear structure of an eigen-value equation for the mass (squared)⁴⁵. The second order formulation of the Dirac equation is easily obtained from (4.5), (4.6) and formally we get

$$\hat{p}_{BA'} \hat{p}^{BB'} \Phi_{B'} = \frac{m^2}{2} \Phi_{A'}, \quad \hat{p}^{AB'} \hat{p}_{BB'} \Psi^B = \frac{m^2}{2} \Psi^A, \quad (4.7)$$

which is equivalent to

$$\begin{pmatrix} 0 & \hat{p}^{CA'} \\ \hat{p}_{AC'} & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{p}^{AB'} \\ \hat{p}_{BA'} & 0 \end{pmatrix} \begin{pmatrix} \Psi^B \\ \Phi_{B'} \end{pmatrix} = \frac{m^2}{2} \begin{pmatrix} \Psi^C \\ \Phi_{C'} \end{pmatrix}. \quad (4.8)$$

We wish to obtain a more physical interpretation of Dirac equation (4.7) (or, equivalently, that in (4.8)). Therefore, we rewrite it as follows

$$\begin{aligned} \hat{p}_{BA'} \hat{p}^{BB'} \Phi_{B'} &= \left[\frac{1}{2} (\hat{p}_{BA'} \hat{p}^{BB'} - \hat{p}_B^{B'} \hat{p}_{A'}^B) + \frac{1}{2} (\hat{p}_{BA'} \hat{p}^{BB'} + \hat{p}_B^{B'} \hat{p}_{A'}^B) \right] \Phi_{B'} \\ &= \frac{m^2}{2} \Phi_{A'}, \end{aligned} \quad (4.9)$$

$$\begin{aligned} \hat{p}^{AB'} \hat{p}_{BB'} \Psi^B &= \left[\frac{1}{2} (\hat{p}^{AB'} \hat{p}_{BB'} - \hat{p}_B^{B'} \hat{p}_{A'}^B) + \frac{1}{2} (\hat{p}^{AB'} \hat{p}_{BB'} + \hat{p}_B^{B'} \hat{p}_{A'}^B) \right] \Psi^B \\ &= \frac{m^2}{2} \Psi^A. \end{aligned} \quad (4.10)$$

After some spinor manipulations, using the Fierz identity in (2.22) and the definition in (4.2), we obtain

$$[\hat{p}_{KK'} \hat{p}^{KK'} \delta_{A'}^{B'} + ie \phi_{A'}^{B'}] \Phi_{B'} = m^2 \Phi_{A'}, \quad (4.11)$$

$$[\hat{p}^{KK'} \hat{p}_{KK'} \delta_B^A - ie \bar{\phi}_B^A] \Psi^B = m^2 \Psi^A, \quad (4.12)$$

where

$$\phi_{A'B'} := \frac{1}{2} \epsilon^{AB} F_{AA'BB'} = \phi_{B'A'} \quad \text{and} \quad \bar{\phi}_{AB} := \frac{1}{2} \bar{\epsilon}^{A'B'} F_{AA'BB'} = \bar{\phi}_{BA}, \quad (4.13)$$

define infinitesimal $\overline{\text{SL}(2, \mathbb{C})}$ and $\text{SL}(2, \mathbb{C})$ transformations, i.e. infinitesimal ‘‘Lorentz’’ transformations of the two spinors at each space-time point. The electro-magnetic field F is defined by the four-potential introduced in (4.1) according to the familiar rule

$$F_{AA'BB'} := \partial_{AA'} A_{BB'} - \partial_{BB'} A_{AA'}. \quad (4.14)$$

⁴⁵which then constitutes a generalisation of the Klein-Gordon equation. The latter describes a non-spinning charged (scalar) relativistic particle in an external electro-magnetic field. Note also that the classical limit of the Klein-Gordon equation reproduces the Lorentz force equation as will be discussed shortly.

For the purpose of physical interpretation we express the electro-magnetic Lorentz tensor field as

$$F_{AA'BB'} := F_{ab}\Sigma_{AA'BB'}^{ab}, \quad (4.15)$$

where $(\Sigma^{ab})_{BB'}^{AA'}$ represents six infinitesimal generators of the Lorentz group acting on the space of mixed spinor-tensors of rank two. However, these generators are composed of two sets of six infinitesimal generators acting on the complex vector space of simple spinors and their complex conjugates. This can be seen very easily because in the usual spinor algebra manner we obtain the decomposition

$$\Sigma_{AA'BB'}^{ab} := \bar{\Sigma}_{A'B'}^{ab}\epsilon_{AB} + \Sigma_{AB}^{ab}\bar{\epsilon}_{A'B'}, \quad (4.16)$$

where the first set of six infinitesimal $\overline{\text{SL}(2, \mathbb{C})}$ transformations (essentially the same as relativistic spin-operator action on \bar{S} (2.1)) and the second set of six infinitesimal $\text{SL}(2, \mathbb{C})$ transformations (essentially the same relativistic spin-operator action on S in (2.1)) are given by

$$\Sigma_{AB}^{ab} := \bar{\epsilon}^{A'B'}\sigma_{AA'}^{[a}\sigma_{BB']}^{b]}, \quad \bar{\Sigma}_{A'B'}^{ab} := \epsilon^{AB}\sigma_{AA'}^{[a}\sigma_{BB']}^{b]}, \quad (4.17)$$

with $\sigma_{AA'}^a$ defining the isomorphism between Lorentz four-vectors and mixed spinor-tensors of second rank. The two dimensional Dirac matrices $\sigma_{AA'}^a$ are also called Infeld-Van der Waerden connecting quantities. This has been explained at the end of section two in this review. We thus have that

$$ie\phi_{A'}^{B'} := \frac{ie}{2}F_{ab}\bar{\Sigma}_{A'}^{abB'} \quad \text{and} \quad -ie\bar{\phi}_B^A := \frac{-ie}{2}F_{ab}\Sigma^{abA}_B, \quad (4.18)$$

which implies that at each space-time point the electro-magnetic field F defines the six infinitesimal parameters of the infinitesimal $\overline{\text{SL}(2, \mathbb{C})}$ and $\text{SL}(2, \mathbb{C})$ transformations acting on each of the two spinors Φ and Ψ separately.

Quantum mechanical operators, i.e. essentially infinitesimal generators of relevant algebras, in the classical limit, become functions on appropriate phase spaces. For example, consider the Klein-Gordon operator

$$\hat{H} := \hat{p}^a\hat{p}_a, \quad (4.19)$$

with \hat{p}_a defined in (4.2). In the classical limit, (4.19) may be regarded as a Poincaré scalar function on the cotangent bundle of the affine Minkowski space T^*M ,

$$\mathcal{H}(P_b, x^a) := (P_b - eA_b(x^a))(P^b - eA^b(x^a)). \quad (4.20)$$

The natural Poincaré invariant symplectic structure on T^*M is then defined by the only non-vanishing set of Poisson brackets

$$\{P_b, x^a\} = \delta_b^a, \quad (4.21)$$

where P_b and x^a are the global Poincaré covariant/invariant canonical coordinates on T^*M . As is well-known, [3], by taking the function in (4.20) as a generator of the canonical flow and projecting this flow onto the Minkowski base space reproduces exactly the Lorentz force equation. The curves of this flow projected onto the Minkowski base space are solutions of the Lorentz force equation.

Encouraged by this fact we were curious about what kind of dynamical relativistic classical equations would appear from such a dequantization of the Dirac equation minimally coupled with an external electro-magnetic field. In other words, we wanted to see what kind of modifications to the Lorentz force equation would be introduced by the classical limit of the electron-like spin (gyromagnetic ratio is automatically equal to 2 for an electron in the Dirac equation). We have now, at our disposal, the classical two-twistor phase space $T\Delta T$. With reference to our discussion above, the classical limit of the six generators of the infinitesimal $\overline{\text{SL}}(2, \mathbb{C})$ and $\text{SL}(2, \mathbb{C})$ transformations should have the intrinsic angular four-momentum function (3.44) on $T\Delta T$ as its classical limit. We make it to our assertion and define the flow generating function on the phase space $T\Delta T$ to be

$$H = H(Z, W, \bar{Z}, \bar{W}) = (P_i - eA_i)(P^i - eA^i) + \frac{1}{2}e\Sigma_{kl}F^{kl}, \quad (4.22)$$

which may be regarded as a classical limit of the minimally coupled second order Dirac operator in (4.11) and (4.12). If spin tensor Σ vanishes, i.e. when the conformal scalars ρ in (3.15) and k in (3.46) are equal to zero, the function in (4.22) coincides with (4.20) and the decomposition of $T\Delta T$ is reduced to the direct sum $T^*M \oplus T^*S^1$. The flow generated in this simplified two-twistor space reproduces again the Lorentz force equation. However, there is a small, and maybe welcome, difference here: the charge comes in as a constant of motion and not just as a number put in by hand. This is a further justification of the definition in (4.22).

On the sixteen dimensional phase space $T\Delta T$ we are only interested in the dynamics of the Poincaré invariant/covariant functions x, P, Σ and e . This makes fourteen variables altogether. There are two additional angle variables, the argument of the Poincaré scalar function f in (3.32) (the absolute value of f is equal to the Minkowski time-like length of the canonical four-momentum P divided by $\sqrt{2}$) which is canonically conjugate to the charge e and also the phase (the flag) of the η spinor canonically conjugate to the conformal real valued scalar k in (3.46). These facts have been discovered in [4, 5]. The generating function in (4.22) is such that the dynamics of these two angles do not mix with the dynamics of the “physical” variables x, P, Σ, e and therefore will not be displayed in what follows. The canonical flow⁴⁶ on the phase space $T\Delta T$ generated by the Poincaré scalar function in (4.22) implies the following equations for the physical variables x, P, Σ, e :

$$\frac{d e}{d \lambda} := \{H, e\} = 0, \quad (4.23)$$

$$\frac{d x^j}{d \lambda} := \{H, x^j\} = 2(P^j - eA^j), \quad (4.24)$$

⁴⁶it is canonical with respect to the symplectic structure defined by the imaginary part of the pseudo-hermitian form preserved by the $SU(2, 2)$ transformations.

$$\frac{dP_j}{d\lambda} := \{H, P_j\} = 2e(P_i - eA_i) \frac{\partial A^i}{\partial x^j} - \frac{1}{2}e\Sigma_{kl} \frac{\partial F^{kl}}{\partial x^j}, \quad (4.25)$$

$$\begin{aligned} \frac{d\Sigma^{ij}}{d\lambda} &:= \{H, \Sigma^{ij}\} = \frac{1}{2}eF_{kl}\{\Sigma^{kl}, \Sigma^{ij}\} = \\ &= \frac{1}{2}eF_{kl}(g^{ik}\Sigma^{lj} - g^{il}\Sigma^{kj} + g^{jl}\Sigma^{ki} - g^{jk}\Sigma^{li}) = \\ &= \frac{1}{2}e(F^i{}_l\Sigma^{lj} - F_k{}^i\Sigma^{kj} + F_k{}^j\Sigma^{ki} - F^j{}_l\Sigma^{li}), \end{aligned} \quad (4.26)$$

where λ labels points along the lines of the flow. It would be more illuminating to have the proper time of the two-twistor particle as an evolution parameter along the flow instead of λ . For notational reasons we first put

$$\Sigma F := \frac{1}{2}\Sigma^{kl}F_{kl}, \quad (4.27)$$

and note that the relation between the proper time τ and the parameter λ reads

$$\left(\frac{d\tau}{d\lambda}\right)^2 := \frac{dx^j}{d\lambda} \frac{dx_j}{d\lambda} = 4(P^j - eA^j)(P_j - eA_j) = 4(H - e\Sigma F)^2, \quad (4.28)$$

where H denotes the constant value of the generating function (trivially, it is a constant of (the flow) motion) that can be identified with the dynamical mass squared of the system. The equation in (4.28) was obtained from (4.24) with additional use of (4.22) where the mechanical four-momentum squared is the difference between the value of H (the dynamical mass squared of the system) and the contribution to this value coming from the interaction of the spinning charge with the external electro-magnetic field. If the dot denotes differentiation with respect to the proper time

$$\dot{x}^j := \frac{dx^j}{d\tau} = \frac{P^j - eA^j}{\sqrt{H - e\Sigma F}}, \quad (4.29)$$

then we obtain

$$\dot{x}^j \dot{x}_j = 1. \quad (4.30)$$

as it should be.

Eliminating the equation of motion for the canonical linear four-momentum, we get finally

$$\frac{de}{d\tau} = 0, \quad (4.31)$$

$$\frac{d^2 x^j}{d\tau^2} = \frac{e}{\sqrt{H - e\Sigma F}} F_i{}^j \frac{dx^i}{d\tau} + \frac{e}{4(H - e\Sigma F)} \left[\frac{\partial F_{kl}}{\partial x^m} \Sigma^{kl} \left(\frac{dx^m}{d\tau} \frac{dx^j}{d\tau} - g^{mj} \right) \right], \quad (4.32)$$

$$\frac{d\Sigma^{ij}}{d\tau} = \frac{e}{4\sqrt{H - e\Sigma F}} (F^i{}_l \Sigma^{lj} - F_k{}^i \Sigma^{kj} + F_k{}^j \Sigma^{ki} - F^j{}_l \Sigma^{li}). \quad (4.33)$$

The equation in (4.32) gives a generalisation of the Lorentz force equation following from the two-twistor dequantization of the minimally coupled Dirac equation. The equation of motion for the (five⁴⁷) spin variables in (4.33), automatically implied by the formalism, differs significantly from the so called BMT (Bargmann, Michel, Telegdi) equation. In the latter, the number of spin variables equals three. Besides, one also requires the norm of the spin four-vector to be a constant of motion. Consequently there are, in total, only two independent variables. Such spin variables have the origin in an, a priori, defined Pauli-Lubański spin four-vector S^a which is constructed as a relativistic generalisation of the non-relativistic spin vector [10, 22]. The so defined S^a has also to fulfill a constraint⁴⁸, $S_a \dot{x}^a = 0$, and the requirement that $-S_a S^a$ is a constant of motion, as was mentioned above.

To our knowledge, the so defined classical BMT equation together with (a number of different suggested versions trying to generalise) the Lorentz force equation so that it also includes spin variables, have never been given a proper relativistic hamiltonian description. The starting point has always been non-relativistic classical mechanics. The relevant discussions concerning these matters may be found in Jackson's, Corben's and Rohrlich's books [10, 7, 22]. There exists also a number of Lagrangian formulations. Some of them make use of the so called anticommuting fermionic numbers and Grassman variables. A relatively recent resume of Lagrangian formulations may be found in [8].

To get our relativistic Hamiltonian formulation, we start with the relativistic Dirac quantum mechanical equation (at the first quantisation level), “classicalise” it on the $T\Delta T$ and thereby obtain the equations in (4.31), (4.32), (4.33). The relativistic Hamiltonian formulation is already there.

Note that from (4.33) it also automatically follows that the conformal scalar defined in (3.49) is a constant of motion.

We do not insist on the fact that the particle should be point-like. On the contrary, for a free two-twistor object two four-positions (one commuting and one non-commuting) were distinguished. The commuting x in (3.40) was identified with the centre of charge while the non-commuting one q in (3.33) was identified with the centre of mass. With the external electro-magnetic field switched on the four position x plays the role of the dynamical variable. The canonical linear momentum four-vector P and the “canonical” four-position of the center of mass q in (3.33) are totally eliminated from the equations of motion. We suggest that the Lorentz space-like four-vector pointing from the centre of charge to a new dynamical centre of mass of the system (now interacting with an external electro-magnetic field) should be defined by

$$\Delta q^i := \frac{1}{\sqrt{H}} \Sigma^{ik} \dot{x}_k, \quad (4.34)$$

⁴⁷one of the Lorentz invariant scalars formed from Σ equals zero, as mentioned before.

⁴⁸very hard if not, in general, impossible to fulfil consistently with the condition that the particle is point-like and described by a Hamiltonian principle.

This coincides with Δx^a in (3.41) in the case when the external electro-magnetic field is zero, i. e. $F_{ab} = 0$. The new dynamical intrinsic angular four-momentum with respect to the centre of charge then reads

$$S^{ij} := \Sigma^{ij} - \sqrt{H}(\dot{x}^i \Delta q^j - \dot{x}^j \Delta q^i) = \Sigma^{ij} - \dot{x}^i \dot{x}_k \Sigma^{jk} + \dot{x}^j \dot{x}_k \Sigma^{ik}, \quad (4.35)$$

and coincides with S^{ij} in (3.37) when $F_{ab} = 0$. The new dynamical Pauli-Lubański spin four-vector S^i reads now as

$$S^i := \frac{\sqrt{H}}{2} \epsilon^{ijkl} \Sigma_{jk} \dot{x}_l, \quad (4.36)$$

and coincides with S^{ij} in (3.38) when $F_{ab} = 0$.

The value of the square of the Lorentz norm of this newly introduced dynamical spin four-vector (S and Δq are space-like Lorentz four-vectors hence the negative signs) is then given by

$$S^2 := \frac{-S_i S^i}{H} = k^2 - \Delta q^i \Delta q_i H, \quad (4.37)$$

and, unlike the value of the function $\Sigma^{ab} \Sigma_{ab} = 4k^2$, is not a constant of motion (except perhaps for some special choices of the external electro-magnetic field); instead we have

$$\frac{dS^2}{d\tau} = -H \frac{d(\Delta q^i \Delta q_i)}{d\tau} \quad (4.38)$$

showing that the value of the square of the norm of the intrinsic (intrinsic with respect to the centre of charge) spin four-vector varies as the square of the distance between the centre of mass and the centre of charge multiplied by the square of the mass of the system. From (4.37) and (4.38) it now follows that if the Pauli-Lubański spin decreases, then the distance between the centre of charge and centre of mass increases and vice versa. Is this a new classical interpretation of the famous “Zitterbewegung”?

Let us now take a look on how the dynamics described above looks like when it is formulated as a minimal action principle on $T\Delta T$. This will take us closer to the, as yet unknown, special relativistic twistor quantum dynamics.

5 AN ACTION PRINCIPLE ON THE TWO-TWISTOR SPACE.

We start with the derivation of the Lorentz force equation from a Poincaré invariant action principle not on the two-twistor space but on T^*M .

The Poincaré invariant symplectic potential defining Poisson bracket structure (which in terms of global canonical Poincaré covariant variables was defined by the canonical Poisson

bracket relations in (4.21)) on T^*M , the (8D) cotangent bundle of the Minkowski space time, is given by⁴⁹

$$\gamma_0 := P_i dx^i, \quad (5.1)$$

where P_i denotes the coordinates of a Lorentz four (co-) vector (i.e. a vector in \mathbb{R}^4 regarded as a covariant tensor of rank one with respect to the Lorentz group), while x^i denotes the coordinates of a Minkowski position four vector (i.e. an affine vector in \mathbb{R}^4 regarded as a contravariant Lorentz four-vector of rank one).

Consider extremal curves (with fixed endpoints) in T^*M of the functional

$$\tilde{S}(\tilde{C}) := \int_{\tilde{C}} \gamma_0, \quad (5.2)$$

where all the curves $\{\tilde{C}\}$ are constrained to lie on the (7D) hypersurface in T^*M defined by a Poincaré invariant condition

$$\tilde{C} \subset \{(x, P) \in T^*M; [P_i - eA_i(x^k)][P^i - eA^i(x^k)] - \mathcal{H} = 0\}, \quad (5.3)$$

with $m^2 := \mathcal{H} > 0$ and e being non-zero constants and with $A_i(x^k)$ being a real Lorentz four (co-) vector valued function on M .

As is well-known, projections of these extremal curves onto the Minkowski space M give space-time trajectories of a charged massive (non-spinning) particle, with fixed charge e and fixed mass m , moving under the action of an external electro-magnetic field defined by the four-potential $A_i(x^k)$. These trajectories are solutions of the so called "Lorentz force equation". The Lorentz force equation itself may also be derived from this principle.

To derive it explicitly and thereby to prove our assertion, one replaces the action in (5.2), subject to the condition in (5.3), by a new action (Lagrange's multiplier method)

$$S(C) := \int_C \left\{ \gamma_0 - \frac{1}{2} l(\lambda) ([P_i - eA_i(x^k)][P^i - eA^i(x^k)] - m^2) d\lambda \right\}, \quad (5.4)$$

where one lets $l(\lambda)$ to be an arbitrary real valued function (Lagrange multiplier to be varied over) of an arbitrary real valued parameter λ labeling points along C , which is now allowed to lie anywhere in T^*M (but with endpoints still fixed at the same positions on the surface defined in (5.3)). The extremal curves of this new action coincide with the extremal curves of the action in (5.2) subject to the condition given by (5.3). Thus, we look after the extremal curves of the functional

$$S(C(\lambda)) := \int_C L(C(\lambda)) d\lambda, \quad (5.5)$$

⁴⁹ T^*M equipped with the symplectic structure Ω_0 defined by γ_0 ($\Omega_0 = d\gamma_0$) is called the extended phase space of a spinless particle; extended because the Poincaré group acts *non*-transitively on T^*M , transitivity being the classical analog of irreducibility.

where (the Lagrangian) $L(C(\lambda))$ on T^*M is given by

$$L(C(\lambda)) = P_i(\lambda) \frac{dx^i}{d\lambda} - \frac{1}{2} l(\lambda) ([P_i(\lambda) - eA_i(x^k(\lambda))] [P^i(\lambda) - eA^i(x^k(\lambda))] - m^2). \quad (5.6)$$

Here $P_i(\lambda)$, $x^i(\lambda)$ and $l(\lambda)$ in (5.6) are regarded as independent (function-) variables of the functional S . Our point is to look for the extremum of S . First, we note that in (5.6) there are no derivatives (with respect to λ) of the functions $P_i(\lambda)$ and of the function $l(\lambda)$. Therefore, the Euler-Lagrange equations with respect to $P_i(\lambda)$ give

$$\frac{dx_i}{d\lambda} - l(\lambda) [P_i(\lambda) - eA_i(x^k(\lambda))] = 0, \quad (5.7)$$

or, equivalently,

$$P_i(\lambda) = \frac{1}{l(\lambda)} \frac{dx_i}{d\lambda} + eA_i(x^k(\lambda)). \quad (5.8)$$

When we plug back this equation into (5.6) (in this way eliminating the variable $P_i(\lambda)$ from (5.6)) yields

$$L(C(\lambda)) = \left(\frac{1}{l(\lambda)} \frac{dx_i}{d\lambda} + eA_i(x^k(\lambda)) \right) \frac{dx^i}{d\lambda} - \frac{1}{2} l(\lambda) \left(\frac{1}{(l(\lambda))^2} \frac{dx^i}{d\lambda} \frac{dx_i}{d\lambda} - m^2 \right), \quad (5.9)$$

or equivalently,

$$L(C(\lambda)) = \frac{1}{2} \left(\frac{1}{l(\lambda)} \frac{dx_i}{d\lambda} \frac{dx^i}{d\lambda} + l(\lambda) m^2 \right) + eA_i(x^k(\lambda)) \frac{dx^i}{d\lambda}. \quad (5.10)$$

The Euler-Lagrange equations with respect to $l(\lambda)$ applied to (5.10) give

$$\frac{1}{2} \left(- \frac{1}{[l(\lambda)]^2} \frac{dx_i}{d\lambda} \frac{dx^i}{d\lambda} + m^2 \right) = 0, \quad (5.11)$$

which yields

$$l(\lambda) = \pm \frac{\sqrt{\frac{dx_i}{d\lambda} \frac{dx^i}{d\lambda}}}{m}. \quad (5.12)$$

By plugging back (5.12) into (5.10) (and thereby eliminating the variable $l(\lambda)$ from (5.10)), we obtain a Lagrangian defined entirely on the (base=configuration) Minkowski space:

$$L(C(\lambda)) = \pm m \sqrt{\frac{dx^i}{d\lambda} \frac{dx_i}{d\lambda}} + eA_i(x^k(\lambda)) \frac{dx^i}{d\lambda}. \quad (5.13)$$

The Lagrangian $L(C(\lambda))$ in (5.13) is, of course (up to the sign ambiguity), the familiar Lagrangian one usually starts with. Its Euler-Lagrange equations, extremalizing the projection (onto the Minkowski space) of the original action given in (5.2) subject to the

condition in (5.3) or equivalently, as given in (5.4), constitute the well-known Lorentz force equation if (in the so arising equations) one chooses the parameter λ to be the proper time of the particle, i.e. if, in the final equation, one requires that $\frac{dx^i}{d\lambda} \frac{dx_i}{d\lambda} = 1$.

Concluding we note that the action principle defined above in (5.2) and (5.3) is equivalent to the method described in the previous section. Therefore we now turn our attention back to the two-twistor phase space $T\Delta T$.

On $T\Delta T$, the (16D) two-twistor space, consider a global symplectic potential (one-form), which is invariant with respect to $SU(2, 2)$ (therefore also essentially⁵⁰ with respect to the Poincaré group), given by $\gamma := \gamma_1 + \gamma_2$, where

$$\gamma_1 := \frac{1}{2}(iZ^\alpha d\bar{Z}_\alpha - i\bar{Z}_\alpha dZ^\alpha) \text{ and } \gamma_2 := \frac{1}{2}(iW^\alpha d\bar{W}_\alpha - i\bar{W}_\alpha dW^\alpha). \quad (5.14)$$

We remind that the symplectic structure $\Omega_i := d\gamma_i$ defined by γ_1 or γ_2 in (5.14) coincides with the imaginary part of the hermitian form (scalar "product") which is preserved by the $SU(2, 2)$ (and therefore (essentially) by the Poincaré group acting on the twistor space T . In effect the Poisson bracket structure defined by $\Omega := d\gamma$ is conformally (and therefore essentially also Poincaré) invariant. In other words, the conformal transformations of $T\Delta T$ are canonical with respect to the symplectic structure Ω . In the previous section we have already used this symplectic structure at the level of the corresponding conformally invariant Poisson bracket algebra.

In the previous section we have proved that the symplectic structure on $T\Delta T$ is decomposed into three Poincaré invariant parts. The Poincaré invariant form of the symplectic potential in (5.14) is then written as follows

$$\gamma = P_i dx^i + ([\sigma^{A'}]d[\eta_{A'}] + [\bar{\sigma}^A]d[\bar{\eta}_A]) + ed\phi. \quad (5.15)$$

The pair of square bracketed spinor coordinates $([\eta_{A'}], [\sigma^{A'}])$ on the real projective spinor space $T^*\mathbb{RP}(S)$ is intended to recall us that these are just equivalence classes with respect to the multiplication (division) by a non zero real number r according to the rule [4, 5]:

$$(\eta_{A'}, \sigma^{A'}) \equiv (r \eta_{A'}, \frac{1}{r} \sigma^{A'}).$$

Recall that P_a , x^a , $\sigma^{A'}$, $\eta_{A'}$, e and $\phi = \arg f$ in (5.15) were defined in (3.35), (3.40), (3.45), (3.28), (3.52) and (3.32). Inserting carefully this chain of definitions into (5.15) and finally using the spinor representation of the two-twistors as in (3.28) reproduce the symplectic potential in (5.14). The reader should perhaps try to perform this spinor algebra manipulations just to convince himself or perhaps find some sign inconsistencies that must be corrected.

Consider now extremal curves $\{\tilde{C}\}$ (with fixed endpoints) in $T\Delta T$ of the functional

$$\tilde{S}_{spin}(\tilde{C}) := \int_{\tilde{C}} \gamma, \quad (5.16)$$

⁵⁰this word indicates that only identity component of the universal covering group of the Poincaré group is considered.

where all these curves $\{\tilde{C}\}$ are required to lie on a (15D) hypersurface in $T\Delta T$ defined by the condition

$$[P_i - eA_i(x^k)][P^i - eA^i(x^k)] + \frac{1}{2}e\Sigma^{ij}F_{ij}(x^k) = H. \quad (5.17)$$

Here $H = \text{const}$, $A_i(x^a)$, as usual, represents the (real) four-potential of an external electromagnetic field $F_{ij}(x^k) = -F_{ji}(x^k)$ acting on the charged massive and spinning particle with its charge equal to the value of function e defined in (3.52) which is a constant of motion.

Without any proof we can now claim that the equations of motion implied by this action principle are exactly the ones described in the previous section in (4.31), (4.32), (4.33). However, this is much harder to prove directly using the action principle in (5.16), (5.17) because the symplectic potential contains the σ and η variables so that they will appear explicitly in the arising equations of motion while we really wish that only Σ variables should appear explicitly (and P , x and e) in the arising equations of motion.

The equations of motion obtained in the previous section must therefore be constructed a posteriori. By the equivalence of the two approaches we know without any calculations that the arising equations for the variables P , x , e and Σ are the same as those in (4.23), (4.24), (4.25) and (4.26).

At the quantum level one is not interested in the equations of motion. The entire action integral is used in quantum mechanics. All paths between two fixed points on the fifteen dimensional surface are of importance, not just the extremal curves. Therefore the above action integral formulation could serve as a starting point to a fresh quantization of the dynamics of a massive, charged and spinning particle thereby generalizing the minimally coupled Dirac equation. The arising equations would then be valid for any quantised value of the spin. Stopping here, we hope that we will be able to come back to this issue in the nearest future.

6 CONCLUSIONS AND REMARKS.

The results presented in this review are not in the main stream of the research within the Twistor Theory. The very ambitious goals of such a research in this main stream led by Roger Penrose, concern the translation of the general relativity theory into the language of holomorphic functions of (many?) twistors. Penrose hopes that this approach will lead to a unification of quantum field theory with the general theory of relativity (of course, describing gravitation in a curved space-time, no longer in the affine Minkowski space) in a way very different from that in the research pursued by the people working with the superstrings, M-theory, etc.. Penrose thinks that both quantum principles and GR must be modified in some way. This way, as Penrose put it, will be pointed out by the many dimensional singularity sets of functions of many twistors and also by means of sophisticated analysis and geometry in multidimensional Riemann-like surfaces defined by these many complex variables holomorphic functions. The singularity surfaces will, according to Penrose, replace the notion of the quantum mechanical Hilbert space of states, simultaneously turning the wave function to an (conformally invariant?) objective entity. This could solve problems with the understanding of the quantum mechanical process of measurement. The gravity effect would then be responsible for the so called

collapse of the wave function. The quantum mechanical superposition principle must then be abandoned (being only the first approximation of the new formulation) for some new and not yet known, but more accurate, principles should replace it. Up to now this goal has been achieved only partially [11, 12, 13].

In this review our goal was very much modest. At the level of special relativistic classical particle dynamics we wanted to see what new elements are brought in if the twistor description was imposed. I think we succeeded quite well. The idea that everything is made of non-local masslessness, inherent in the twistor formalism from the very start by the built-in conformal symmetry, seems to be very appealing. The holomorphic aspects are not so important at the level of classical relativistic twistor dynamics. Instead, the symplectic structure (the imaginary part of the pseudo-hermitian form preserved by $SU(2,2)$) is emerging as the most important element of such a twistor approach. This is not so surprising because a conformally invariant first quantization leads automatically to the aspects of holomorphy [16, 19].

We think that the classical equations of motion of the type described in this review are worthwhile to investigate. It would be interesting to explore some possible consequences of the obtained equations of motion above. For that reason we should reintroduce the Planck constant \hbar and the velocity of light constant c into the obtained equations and investigate the non-relativistic limit to some order of $(\frac{\sqrt{\dot{x}^a \dot{x}_a}}{c})^n$ and some order of $(\frac{\hbar}{k})^m$. Once this is done, it would be of outmost importance to recast the so obtained approximations into a new non-relativistic hamiltonian formulation containing relativistic and spin induced correction terms coming from our approach. Such approximative models should have predictive power and perhaps some experimental proposals could emerge to test them. These concrete calculations are left for the future while we hope to be able to perform them in cooperation with the members of the research group of Professor Iouri Mikhailovich Vorobiev who are experts in this domain.

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